Approximate methods for solving differential fractional equations

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Abstract

It is remarkable that the make of explicit sequence of logic has, if not the growth, at least the age of a usual differential gear of sequence of logic that was given birth by the work of British Isaac Newton and of the German G. Leibniz. It is known that Leibniz used first the symbolism $\frac{d^n y}{dx^n}$ *for an n-order derivative of a function* $y(x)$*. This was perhaps an unsophisticated game of symbols that made L'Hospital to ask Leibniz (with personal correspondence in 1695) what would happen in the case where n was a fraction. "What happens when n is* $\frac{1}{2}$*" asked L'Hospital. The insufficient but prophetic answer of Leibniz was: "This would lead in a certain paradoxical. From this paradoxical, however, we will come out one day with very useful consequences". The first time where the significance of derivative of an accidental order on the differential and absolute sequence of logic was presented in 1819 in the book of P. F. Lacroix. Lacroix dedicates less than two pages in this subject, his book, however, is constituted by 700 pages. Beginning from the* $y(x) = x^n(1)$ *where n is a positive entire number, he showed that the m-order of a derivative function* $y(x)$ *was the* $\frac{d^m y}{dx^m} = \frac{n!}{(n-m)!}x^{n-m}$ *(2). Then,by using the symbol of Legendre Gamma (*Γ *that is to say* Γ*-function),* $\frac{d^{1/2}y}{dx^{1/2}} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)}x^{\alpha-1/2}$ *(3). That generalises the growth of factors, and replacing m with* $\frac{1}{2}$ *and n with any positive real number* α*, Lacroix "manufactured" the type (3) that expresses the derivative of order of* $\frac{1}{2}$ *of a function. We remind to the reader that the* $\Gamma-$ *function is fixed with the each integral* $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ *(4) for a number of z in the transcendental level apart from the points* $z \neq 0, -1, -2, -3, \ldots$ *Certain of the basic attributes of Gamma (*Γ *function is the function* $\Gamma(z+1) = z\Gamma(z)$ *(5), the function* $\Gamma(n+1) = n!$ *(6) as well as the function* $\Gamma(1) = 1 + \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$ *(7). Type (3) was a formal product of the way with which worked the mathematicians of that period. Making use of relations (5)-(7) Lacroix calculated, as an application, the derivative of order* $\frac{1}{2}$ *of the function* $y(x) = x$ *and took the function* $\frac{d^{1/2}x}{dx^{1/2}} = \frac{2}{\sqrt{\pi}} \cdot \sqrt{x}$ *(8). This value of this half derivative of x that answers in the interrogative proposal of the title of this article, coincides with the corresponding value that gives the admissible today definition of an explicit derivative at Riemann-Liouville. It deserves to be stressed in this article that since then that L'Hospital placed the question of the explicit derivative (1695), needed to pass 124 years in order to be presented a formal answer in the book of Lacroix (1819), and 289 years in order to be written the first book that was dedicated in this subject (1974). Euler and Fourier even if they cited derivative explicit of order, did not give any application neither any example. Thus, the value for the first application of significance of explicit derivative belongs to Niels Henrik Abel, which applied in 1823 the explicit sequence of logic in solution of an absolute equation that is presented in the study of problem of simultaneous time. This problem, that sometimes is also named problem of equal time, is reported in the finding of form of orbit that is found on a vertical level, so that the time that is needed a material point in order to slip up to the more inferior point of orbit, is independent from the point of starting line of movement. In 1850 William Center observed that the difference between these two definitions of explicit derivative was found in the value that was predicted for the explicit derivative of a constant factor of a function. According to the definition of Lacroix, the explicit derivative constant factor of a function was not annihilated, while according to the definition (9) of Liouville each explicit derivative constant factor of function was zero because* $\Gamma(0) = \infty$ *(9). In 1847 Riemann followed a way in order to fix the functional (explicit) integral. Supported in the observation of Liouville, according to which the general solution of the equation*

$d^n y(x)/dx^n = 0$ (10) is the equation

$$ {}_cD_x^v f(x) = {}_cD_x^m \, {}_cD_x^{m-v} f(x) = \frac{d^m}{dx^m} \frac{1}{\Gamma(m-v)} \int_c^x (x-t)^{m-v-1} f(t) dt, $$

that is also generalised for an n accidental positive number, attempted to fix explicit integration with type $D^{-v} f(x) = \frac{1}{\Gamma(v)} \int_0^\infty (x-t)^{v-1} f(t) dt + \Psi(x)$ (12). Where v is an accidental positive number and the integral is emanated from the type of Cauchy for multiple integrals, but also the function $\Psi(x)$ is a function of integration (no anymore constant factor of integration) for which it will be in effect that $D' \Psi(x) = 0$ (13), that is to say Ψ is the solution of equation (10) with $n = v$. Cauchy observed in 1880 that also the explicit integral of Riemann (12) had a form that was not determined. The development, however, of the mathematic ideas, seldom becomes without faults. Thus, therefore, also in explicit sequence of logic the growth of significances followed the path of faults, even if in their way met mathematicians of the class of Liouville and Riemann. All above paradoxical today have been untied and has been also found the cause of their creation that is owed basically in that no one of these mathematicians was thought to ask the consequences of all above definitions in the transcendental level. In this article we will give the basic definitions of derivative growth and integration of a random real order first and a random transcendental order second, and also give a brief analysis of the meanings. We will end by attaching some several examples and applications of explicit logic which also involve the solution of such equations. We will also find short ways for approximate solution of differential fractions and software for complete differential equations.

Part.I General Theory

I. Fractional integration (Riemann)

Riemann tried to approach fractional calculus through the process of integration, i.e. the representation of integration in a way that allows the extension of the order of integration beyond the natural numbers.

This approach was as follows.For any natural number n the n-th integral of the function f in the interval $[0, x]$ is given by

$$ D^{-n} f(x) = \int_0^x \int_0^{t_n} \int_0^{t_{n-1}} \dots \int_0^{t_2} f(t) dt_1 dt_2 \dots dt_{n-1} dt_n + $$

$$ + c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_{n-1} x + c_n \tag{1.I} $$

Where D^{-n} symbolizes n successive integrations. But we know from the classical integral calculus that

$$ \int_0^x \int_0^{t_n} \int_0^{t_n-1} \dots \int_0^{t_2} f(t) dt_1 dt_2 \dots dt_{n-1} dt_n = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt \tag{2.I} $$

Therefore the relationship (1.I) is done at

$$ D^{-n} f(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt + \sum_{k=1}^{n} c_k x^{n-k} \tag{3.I} $$

More specifically, assuming all integration constants are zero, (3.I) is written

$$ D^{-n} f(x) = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t) dt \tag{4.I} $$

The right-hand side of (I.4) has meaning also in the real so we can make an extension from the physical to the real for every positive real n. Therefore we define the p > 0 order of integration of f by the formula

$$D^{-p}f(x) = \frac{1}{\Gamma(p)}\int_0^x (x-t)^{p-1}f(t)dt$$

$$\text{if } f(0) = f^1(O) = \ldots = f^{[p]-1}(O) = O \tag{5.I}$$

where [p] denotes the integer part of p. It is obvious that type (5.I) cannot be extended for p < 0, which would imply fractional derivation because then p − 1 < −1 and the integrable function would exhibit a nonintegrable singularity in the upper limit of the integral $t = x$.

Example I.1.

For $p \in (0, 1)$ we have

$$D^{-p}f(x) = \frac{1}{\Gamma(p)}\int_0^x (x-t)^{p-1}f(t)dt = -\frac{1}{p\Gamma(p)}\int_0^x d(x-t)^p t =$$

$$= -\frac{1}{\Gamma(p+1)}\left[t(x-t)^p t|_{t=0}^{t=x} - \int_0^x (x-t)^p dt\right] = \frac{x^{p+1}}{(p+1)\Gamma(p+1)} =$$

$$= \frac{x^{p+1}}{\Gamma(p+2)}$$

The limiting values of p give

$$\lim_{p\to 0^+} \frac{x^{p+1}}{\Gamma(p+2)} = \frac{x}{\Gamma(2)} = x \quad \text{and} \quad \lim_{p\to 1^-} \frac{x^{p+1}}{\Gamma(p+2)} = \frac{x^2}{\Gamma(3)} = \frac{x^2}{2\Gamma(2)} = \frac{x^2}{2}$$

coinciding with the classical analogue of the identity operator and integration.

Therefore the effect of the fractional operator $D^p, p \in R$ on the monomial x results in the exponent changing from 1 to 1 + p.

II. Fractional derivation

Given the fact that derivation and integration are inverse infinite processes. Thus, assuming that we want to calculate the derivative of the $D^p f(x)$, $p \in R$ we can apply the integral operator $D^{-1-p)}f(x), p \in R$ and then obtain the classical first-order derivative of the function $D^{p-1}f(x)$. So we will have

$$D^1 D^{p-1}f(x) = D^{1+p-1}f(x) = D^p f(x), p \in R \tag{1.II}$$

In general the p-order derivative of f with $p > 0$ in R is defined below. We set $m = [p] + 1$ so that $m - p \in (0, 1)$ and the definition of the p-order derivative takes the form

$$D^p f(x) = D^m D^{-(m-p)} f(x) =$$

$$= \frac{d^m}{dx^m}\frac{1}{\Gamma(m-p)}\int_0^x (x-t)^{m-p-1}f(t)dt \tag{2.II}$$

II.1. The rule of Leibnitz

We know Leibnitz's rule for the derivation of integrals that

$$\frac{d}{dx}\int_{f_1(x)}^{f_2(x)} g(x,t)dt = g\left(x, f_2(x)\right) f_2'(x) - g\left(x, f_1(x)\right) f_1'(x) + \int_{f_1(x)}^{f_2(x)} \frac{\partial}{\partial_x} g(x,t)dt \tag{3.II}$$

The difficulty to apply the formula directly is negated if we perform the integration first. This is shown in the example below

Example II.1.

For $p = 1/2$ we have

$$D^{1/2}c = D^1 D^{-1/2}c = \frac{1}{\Gamma(1/2)} D^1 \int_0^x (x-t)^{1-1/2-1} c dt = -\frac{1}{p\Gamma(p)} \int_0^x d(x-t)^p t =$$

$$\frac{c}{\sqrt{\pi}} D^1 \int_0^x (x-t)^{-1/2} dt = \frac{x^{p+1}}{\Gamma(p+2)} = \frac{2c}{\sqrt{\pi}} D^1 x^{1/2} = \frac{c}{\sqrt{\pi}} x^{-1/2}$$

Especially apply

$$D^{1/2}1 = \frac{1}{\sqrt{\pi}} x^{-1/2}$$

$$D^{-1/2}1 = \frac{2}{\sqrt{\pi}} x^{1/2}$$

Based on the above we can find the fractional 1/2 order derivative of the function $x^{3/2}$.

Example II.2.

We will therefore have

$$D^{1/2}x^{3/2} = D^1 D^{-1/2} x^{3/2} = D^1 \frac{1}{\Gamma(1/2)} \int_0^x (x-t)^{1/2-1} t^{3/2} dt$$

$$\frac{1}{\sqrt{\pi}} D^1 \frac{1}{\Gamma(1/2)} \int_0^x t\sqrt{t}(x-t)^{1/2-1} dt = \frac{-2}{\sqrt{\pi}} D^1 \int_0^x t\sqrt{t} d(x-t)^{1/2} =$$

$$= \frac{3}{\sqrt{\pi}} D^1 \int_0^x \left(xt - t^2\right)^{1/2} dt = \frac{3}{\sqrt{\pi}} D^1 \left[\frac{2t-x}{4}\sqrt{xt-t^2} + \frac{x^2}{8} \operatorname{Sin}^{-1} \frac{2t-x}{4}\right]\Bigg|_{t=0}^{t=x}$$

$$= \frac{3\sqrt{\pi}}{8} D^1 x^2 = \frac{3\sqrt{\pi}}{4} x$$

Example II.3

In the case of growth of derivatives of non-explicit order, we have

$$ {}_0D_x^{\sqrt{3}}x^\rho = {}_0D_x^2 D_x^{-(2-\sqrt{3})}x = \frac{d^2}{dx^2}\frac{1}{\Gamma(2-\sqrt{3})}\int_0^x (x-t)^{2-\sqrt{3}-1}tdt = \int_0^x (x-t)^{2-\sqrt{3}-1}tdt = $$

$$ \frac{d^2}{dx^2}\frac{1}{\Gamma(2-\sqrt{3})}\left[\frac{(x-t)^{3-\sqrt{3}}}{3-\sqrt{3}} - x\cdot\frac{(x-t)^{2-\sqrt{3}}}{2-\sqrt{3}}\right]_{t=0}^{t=x} = \frac{1}{\Gamma(2-\sqrt{3})}\left(\frac{1}{2-\sqrt{3}} - \frac{1}{3-\sqrt{3}}\right)\frac{d^2}{dx^2}x^{3-\sqrt{3}} = $$

$$ \frac{1}{\Gamma(2-\sqrt{3})}x^{1-\sqrt{3}} $$

II.2. Complex order operators

In the case where the class of the operator is complex with $p = \operatorname{Re} p + i \operatorname{Im} p$. Then for the existence of the integral

$$ D^{-P}f(x) = \frac{1}{\Gamma(p)}\int_0^x (x-t)^{p-1}f(t)dt \tag{4.II} $$

Required $\operatorname{Re} p > 0$ condition ensuring the existence of the integral. Home we transform the integral through the transformation

$$ t = x\left(1 - e^{-\varphi}\right) $$

$$ dt = xe^{-\varphi}d\varphi $$

We take then

$$ D^{-P}f(x) = \frac{1}{\Gamma(p)}\int_0^{+\infty} x^{p-1}e^{-(p-1)\varphi}f(\varphi)xe^{-\varphi}d\varphi = $$

$$ \frac{x^p}{\Gamma(p)}\int_0^{+\infty} e^{-p\varphi}f(\varphi)d\varphi = \frac{x^p}{\Gamma(p)}L\{f\}(p) \tag{5.II} $$

Where

$L\{f\}(p) = \int_0^{+\infty} e^{-p\varphi}f(\varphi)d\varphi$ is the transformation Laplace of f function. Fractional integration is therefore associated with the Laplace transformation. If in relation (5.II) we consider $p = -i\lambda$ assuming the existence of the integral we will have

$$ D^{-p}f(x) = \frac{x^{-i\lambda}}{\Gamma(-i\lambda)}\int_0^{+\infty} e^{i\lambda\varphi}f(\varphi)d\varphi = $$

$$ = \frac{x^{-i\lambda}}{\Gamma(-i\lambda)}\sqrt{\frac{\pi}{2}}\left(\sqrt{\frac{2}{\pi}}\left(\int_0^{+\infty}\cos(\lambda\varphi)f(\varphi)d\varphi + \sqrt{\frac{2}{\pi}}\left(\int_0^{+\infty}\sin(\lambda\varphi)f(\varphi)d\varphi\right)\right.\right. $$

using the Fourier transformation. Finally, the condition for p can be extended to $\operatorname{Re} p \geq 0$.

Example II.4

We consider the class integration $p = p_1 + \dot{p}_2$ of function x^a to which applies $x^a = 0$ if $x \leq 0$.

$$D^{-p}x^a = \frac{1}{\Gamma(p)} \int_0^x (x-t)^{g-1} t^a dt = \frac{x^{p+a}}{\Gamma(p)} \int_0^1 \left(1 - \frac{t}{x}\right)^{p-1} \left(\frac{t}{x}\right)^{a+1-1} d\left(\frac{t}{x}\right)$$

$$= \frac{x^{p+a}}{\Gamma(p)} B(p, a+1) = \frac{x^{p+a}}{\Gamma(p)} \frac{\Gamma(p)\Gamma(\alpha+1)}{\Gamma(p+a+1)} =$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(p+a+1)} x^{p+a} \tag{7.II}$$

with the conditions $\alpha \neq -1$ and $a+p \neq 1$. Both Γ and B function have been extended to the whole complex plane with the exception of negative integers and zero. More generally, the type (7.II) is written

$$D^{-P}x^a = \frac{\Gamma(\alpha+1)}{\Gamma(p+a+1)} x^{p_1+ip_2+a_1+ia_2} =$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(p+a+1)} x^{\mathrm{a}+a_1} \left[\cos\left((p_2+a_2)\ln x\right) + i\sin\left((p_2+a_2)\ln x\right)\right] \tag{8.II}$$

especially for real values of p, a we will have that

$$\cos\left((p_2+a_2)\ln x\right) + i\sin\left((p_2+a_2)\ln x\right)] = 1$$

Also apply here special $D^{-p}D^{-q}x^a = D^{-q}D^{-p}x^a$.

For the fractional derivative of order p in C, with Re(p) $>$ 0 defined according to the corresponding real order If we have a function f, for which f(x) = 0 for xε = 0 and m = [Re(p)] + 1, then we define the p derivative of f by the formula

$$D^p f(x) = D^m D^{-(m-p)} f(x)$$

In fractional calculus, therefore, we see that every extension from real orders of integration to complex ones is achieved by keeping the same formulas, but with attention to the importance of the parameters.

II.3 General form of Classical operators

A function is analytic at the zero point when the following conditions hold:

i) There are all the derivatives $f^n(0)$, $n = 1, 2, 3, \ldots$

ii) The series $\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$ converges to an interval $(-R, +R)$ and defines the function $g(x)$

iii) $g(x) = f(x), x \in (-R, +R)$

The convergence of the series in condition ii.is, absolute and almost uniform. ie. it is uniform in every compact and closed and blocked subset of the convergence interval $(-R, +R)$,therefore we can derivation and integration this series as many times as we want.

More specifically for p > 0 we will have for the integration

$$D^{-p}\sum_{n=0}^{\infty}\frac{f^n(\mathrm{O})}{n!}x^n=\sum_{n=0}^{\infty}\frac{f^n(\mathrm{O})}{n!}D^{-p}x^n=$$

$$=\sum_{n=0}^{\infty}\frac{f^n(\mathrm{O})}{n!}\frac{\Gamma(n+1)}{\Gamma(n+1+p)}x^{n+p}=\sum_{n=0}^{\infty}\frac{f^n(\mathrm{O})}{\Gamma(n+1+p)}x^{n+p} \tag{9.II}$$

Similarly for the p-order derivative of $f, m<p<m+1$ we get

$$D^{p}\sum_{n=0}^{\infty}\frac{f^n(\mathrm{O})}{n!}x^n=\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}D^{p}x^n=$$

$$=\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}\frac{\Gamma(n+1)}{\Gamma(n+1-p)}x^{n-p}=\sum_{n=0}^{\infty}\frac{f^n(0)}{\Gamma(n+1-p)}x^{n-p} \tag{10.II}$$

We can therefore integrate and derivative in any order, any analytic function, within the analyticity space of.

III. Certain applications

In this last part of article we will try to give certain applications of explicit sequence of logic, for three mainly reasons. Firstly, for the creation of motives for a deeper research of this sector, secondly, in order to give a limited description of the way of confrontation of the corresponding problems, and finally, in order to be vindicated the existence explicit reasonable as self-existent mathematic sector.

Example III.1

A first example is reported in the calculation of not elementary certain integrals.

$${}_cD_x^{-q}f(x)=\frac{1}{\Gamma(q)}\int_c^x(x-t)^{q-1}f(t)dt, q>0 \tag{1.III}$$

with the transformation

$$t=x-x\lambda \tag{2.III}$$

From the relation (1.III) and with the transformation (2.III) we take the type (3.III), and for $x=1$ the type (46).

$${}_0D_x^{-q}f(x)=\frac{1}{\Gamma(q)}\int_0^x(x\lambda)^{q-1}f(x-x\lambda)xd\lambda \tag{3.III}$$

And for $x=1$ we have

$$ {}_0D_1^{-q} f(x) = \frac{1}{q\Gamma(q)} \int_0^1 f(1-\lambda) d\lambda^q \tag{4.III} $$

Placing $\lambda^q = z, p = \frac{1}{q}$ (5.III) the type (4.III) is also written as type

$$ \int_0^1 f\left(1 - z^p\right) dz = \Gamma\left(\frac{p+1}{p}\right)_0 D_1^{-1/p} f(x) \tag{6.III} $$

The type (6.III) is very useful for the calculation of integrals as the type (7.III) or the type (8.III).

$$ \int_0^1 e^{1-z^{2/3}} dz = \Gamma\left(\frac{5}{2}\right)_0 D_1^{-1/2} e^x = 1.5451 \tag{7.III} $$

$$ \int_0^1 \sin\sqrt{1-z^2} dz = \Gamma\left(\frac{3}{2}\right) {}_0D_1^{-1/2} \sin\sqrt{x} = 0.69123 \tag{8.III} $$

A second example is reported in the finding of a simple way for the solution of certain absolute equations of type of Volterra. We consider for example the problem of finding the function $f(x)$ that would satisfy the equation

$$ xf(x) = \int_0^x (x-t)^{-1/2} f(t) dt = \Gamma\left(\frac{\pi}{2}\right) D^{-1/2} f(x) \tag{9.III} $$

The absolute equation (9.III) is also a type of Volterra of third class with irregular core, that is to say an equation that without fail cannot be characterized of the simplest. Explicit sequence of logic, however, makes the solution really simple. We observe that from the rule of explicit integral we have the type

$$ xf(x) = \Gamma\left(\frac{1}{2}\right) {}_0D_x^{-1/2} f(x) \tag{10.III} $$

Symbolizing the ${}_0D_x^{-1/2}$ with $D^{-1/2}$ for simplification we have the type (11.III) and taking the mathematic derivative of order $\frac{1}{2}$ of the two members of type (11.III), it results to the type (12.III).

$$ xf(x) = \sqrt{\pi}_0 D_x^{-1/2} f(x) \tag{11.III} $$

and we take if from both sides the derivative 1/2

$$ D_x^{1/2} xf(x) = \sqrt{\pi}_0 D_x^{1/2} {}_0D_x^{-1/2} f(x) = \sqrt{\pi} f(x) \tag{12.III} $$

According to the rule of Leibniz we have the type (13.III), and the type (12.III) is written as type (14.III) because derivative's superior orders x are annihilated. We replace the type (53) in the type (14.III) and we have the type (15.III).

$$ {}_0D_x^v(fg) = \sum_{n=0}^{\infty} \binom{v}{n} \left({}_0D_x^n f\right)\left({}_0D_x^{v-n} g\right) \tag{13.III} $$

the (12.III) is written

$$ xD_x^{1/2} f(x) + \frac{1}{2} {}_0D_x^{-1/2} f(x) = \sqrt{\pi} f(x) \tag{14.III} $$

because the higher order derivatives are zeroed. From (11.III&14.III) we will have

$$xD_x^{1/2}f(x) + \frac{x}{2\pi}f(x) = \sqrt{\pi}f(x) \tag{15.III}$$

We can however calculate the function $D^{1/2}f(x)$ by taking its derivative's first class of the type (53), and then we are resulted to the type (16.III) or the type (17.III).

$$D[xf(x)] = \sqrt{\pi}D^{1/2}f(x) \text{ (16. III) and then } -xf'(x) + f(x) = \sqrt{\pi}D^{1/2}f(x) \tag{17.III}$$

Erasing then the function $D^{1/2}f(x)$ from the types (15.III) and (17.III) we are leaded to the usual differential equation

$$x^2f'(x) + \left(\frac{3x}{2} - \pi\right)f(x) = 0 \tag{18.III}$$

This equation has its solution as the type $f(x) = cx^{-3/2}e^{-\pi/x}$ (19.III)

Where c is a regularly integration. The function (19.III) constitutes the solution of an irregular absolute equation (9.III). We note here that the difficulty of solution of the equation (9.III) is owed in the non-explicit character of the core $(x-t)^{-1/2}$ and that the accountant profit from the use of explicit sequence of logic is found in the make that the symbolism of derivative of order $\frac{1}{2}$ changes the character of core in explicit number.

Example III.2

We will analyze two examples with Transcendental derivative and integration

i) For integration we will calculate the π-order integral of the function x^e, will we have:

$$\begin{aligned} D^{-\pi}x^e &= \frac{1}{\Gamma(\pi)}\int_0^x (x-t)^{\pi-1}t^e dt = \frac{x^{\pi+e}}{\Gamma(\pi)}\int_0^1 \left(1-\frac{t}{x}\right)^{\pi-1}\left(\frac{t}{x}\right)^{e+1-1} d\left(\frac{t}{x}\right) \\ &= \frac{x^{\pi+e}}{\Gamma(\pi)}B(\pi, e+1) = \frac{x^{\pi+e}}{\Gamma(\pi)}\frac{\Gamma(\pi)\Gamma(e+1)}{\Gamma(\pi+e+1)} = \\ &= \frac{\Gamma(e+1)}{\Gamma(\pi+e+1)}x^{\pi+e} \end{aligned}$$

ii) From II. 2 relationship and for derivative we will calculate the π-order derivation of the function x^e, will we have:

$$\begin{aligned} D^{\pi}x^e &= \frac{d^4}{dx^4}\frac{1}{\Gamma(4-\pi)}\int_0^x (x-t)^{4-p-1}t^e dt = \frac{d^4}{dx^4}\frac{x^{4-\pi+e}}{\Gamma(4-\pi)}\int_0^1 \left(1-\frac{t}{x}\right)^{(4-\pi)-1}\left(\frac{t}{x}\right)^{e+1-1} d\left(\frac{t}{x}\right) \\ &= \frac{1}{\Gamma(4-\pi)}\frac{d^4}{dx^4}x^{4-\pi+e}B(4-\pi, e+1) = \frac{1}{\Gamma(4-\pi)}\frac{d^4}{dx^4}x^{4-\pi+e}\frac{\Gamma(4-\pi)\Gamma(e+1)}{\Gamma(5+e-\pi)} = \\ &= x^{e-\pi}\frac{\Gamma(e+1)}{\Gamma(5+e-\pi)}(4-\pi+e)(3-\pi+e)(2-\pi+e)(1-\pi+e) = \frac{\Gamma(e+1)}{\Gamma(e-\pi+1)}x^{e-\pi} \end{aligned}$$

4 follows because $[\pi] = 3 < \pi < [\pi]+1 = 4$.

Example III.3

The following problem, admittedly a bit contrived ,serves the purposes of showing how the kernel function of the form $(x-t)^v$ and the integral equation is formuleted in the physical sciences. The problem is to determine the shape f(y) of a weir notch, ab opening in a dam, in which the volume Flow rate of fluid, Q,through the notch is expressed as a function of the height h as figure 1.

Apply the equation $dQ = v_y dA$ and $v_y = (2g)^{1/2}(h-y)^{1/2}$ (a.III). But apply for the function of the intersection of the figure that $dA = 2f(y)dy => dQ = 2(2g)^{1/2}(h-y)^{1/2}f(y)dy$ (b.III).

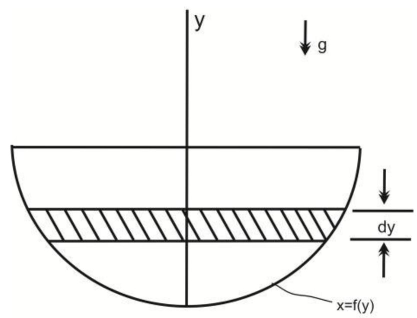

Figure 1:

Because the relationship finally applies

$$Q(h) = c\int_0^h (h-y)^{1/2} f(y)dy = c\Gamma\left(\frac{3}{2}\right) D_h^{-3/2} f(h) \tag{c.III}$$

and

$$f(h) = \frac{1}{c\Gamma\left(\frac{3}{2}\right)} D_h^{3/2} Q(h) = \frac{1}{c\Gamma\left(\frac{3}{2}\right)} \frac{d^2}{dh^2} \frac{1}{\Gamma\left(\frac{1}{2}\right)^0} \int_0^h (h-y)^{-1/2} g(y)dy. \tag{d.III}$$

In case that we have $g(y) = Q(h)/c = h^a$ then we have the relation

$$g(y) = h^a \Rightarrow f(h) = \frac{2}{\pi^{1/2}} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)} h^{a-3/2}. \tag{e.III}$$

Part.II Special theory of solution of fractional differential equations

IV. Grunwald-Letnikov theorie - Derivation and integration of real order

If we extend the solution from integers to real values of order p, we define the integral of order $p > 0$, with formula

$$
\begin{aligned}
&{}_\gamma D_x^{-p} f(t) = \lim_{\substack{h\to 0\\ nh=t-\gamma}} h^p \sum_{r=0}^{n} \left[\begin{array}{c} p \\ r \end{array} \right] f(t - r\cdot h),\ p>0 \\
&\frac{1}{\Gamma(p)} \int_\gamma^t (t-\tau)^{p-1} f(\tau) d\tau
\end{aligned}
\tag{IV.1}
$$

Where

$$
\left[\begin{array}{c} p \\ r \end{array} \right] = \frac{p(p+1)\cdots(p+r-1)}{r!} = (-1)^r \left(\begin{array}{c} -p \\ r \end{array} \right)
$$

Respectively, we define the derivative of order p

$$
\begin{aligned}
&{}_\gamma D_x^{p} f(t) = \lim_{\substack{h\to 0\\ nh=t-\gamma}} \frac{1}{h^p} \sum_{r=0}^{n} (-1)^r \left(\begin{array}{c} p \\ r \end{array} \right) f(t - r\cdot h),\ p>0 \\
&\frac{d^m}{dt^m} \frac{1}{\Gamma(m-p)} \int_\gamma^t (t-\tau)^{m-p-1} f(\tau) d\tau
\end{aligned}
\tag{IV.2}
$$

Where $m = [p] + 1$

If the function ϕ has continuous products up to and including the integer order n, then by successive factorial integrations the previous formulas become

$$
{}_\gamma D_x^{-p} f(t) = \sum_{k=0}^{n-1} \frac{f^k(\gamma)(t-\gamma)^{p+k}}{\Gamma(p+k+1)} + \frac{1}{\Gamma(p+n)} \int_\gamma^t (t-\tau)^{p+n-1} f(\tau) d\tau, n \in N,\ p<n \tag{IV.3}
$$

And also

$$
{}_\gamma D_t^{p} f(t) = \sum_{k=0}^{n-1} \frac{f^k(\gamma)(t-\gamma)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+n)} \int_\gamma^t (t-\tau)^{-p+n-1} f^n(\tau) d\tau, n \in N,\ p<n \tag{IV.4}
$$

Since f is continuous and has n derivatives.

V. Fractional derivative of integral with parameter

In classical calculus we have

$$
\frac{d}{dt} \int_0^t F(t,\tau) d\tau = \int_0^t \frac{\partial}{\partial t} F(t,\tau) d\tau + \lim_{\tau \to t-} F(t,\tau) \tag{V.1}
$$

We will show that the corresponding formula in fractional calculus takes the form

$$
{}_0 D_t^p \int_0^t K(t,\tau) d\tau = \int_0^t {}_0 D_t^p K(t,\tau) d\tau + \lim_{\tau \to t^-} D_t^{p-1} K(t,\tau),\ 0<p<1 \tag{V.2}
$$

Proof

We have

$$
\begin{aligned}
{}_0D_t^p \int_0^t K(t,\tau)d\tau =& \frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t (t-h)^{-p}\int_0^t K(h,\tau)d\tau dh = \\
& \frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t \int_\tau^t (t-h)^{-p} K(h,\tau) dh d\tau = \\
& \frac{d}{dt}\int_0^t K(t,\tau)d\tau = \int_0^t \frac{\partial}{\partial t} K(t,\tau)d\tau + \lim_{\tau\to t^-} K(t,\tau)
\end{aligned}
$$

Where

$$
\begin{aligned}
{}_0D_t^p \int_0^t K(t,\tau)d\tau =& \frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t (t-h)^{-p}\int_0^t K(h,\tau)d\tau dh = \\
& \frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t \int_\tau^t (t-h)^{-p} K(h,\tau) dh d\tau
\end{aligned}
$$

The above proof leads to the useful formula

$$
{}_0D_t^p \int_0^t K(t-\tau)f(\tau)d\tau = \int_0^t (t-\tau)\left[{}_0D_\tau^p K(\tau)\right] + \lim_{\tau\to t^-} f(t-\tau)\left[{}_0D_\tau^{p-1}K(\tau)\right], p \in (0,1) \tag{V.3}
$$

Example.V-1

In the convolution formula V.3 we consider the functions

$$
K(t) = t^2, f(t) = t
$$

Answer: Where we will have

$$
\begin{aligned}
\int_0^t K(t-\tau)f(\tau)d\tau = \int_0^t (t-\tau)^2\tau d\tau = \int_0^t \left(t^2\tau - 2\tau^2 t + \tau^3\right)d\tau = \\
\frac{\tau^4}{2} - 2\frac{\tau^4}{3} + \frac{\tau^4}{4} = \frac{\tau^4}{12}
\end{aligned}
\tag{V.4}
$$

then we take the formula

${}_0D_t^p t^v = \frac{\Gamma(v+1)}{\Gamma(v+1-p)} t^{v-p}$ valid for $p < 0$ and $v > -1$ or for $n > m$ and $0 \le m \le p < m+1$.

Will be take

$$
{}_0D_t^p \frac{t^4}{12} = \frac{\Gamma(5)}{12\cdot\Gamma(5-p)} t^{4-p}
$$

$$
{}_0D_\tau^p K(\tau) = {}_0D_\tau^p \tau^2 = \frac{\Gamma(3)}{\Gamma(3-p)} t^{2-p}
$$

$$
{}_0D_\tau^{p-1} K(\tau) = {}_0D_\tau^{p-1} \tau^2 = \frac{\Gamma(3)}{\Gamma(4-p)} t^{3-p}
$$

The 1st and 2nd part of the general equation V.3 will be

$${}_0D_t^p \int_0^t K(t-\tau)f(\tau)d\tau = \frac{\Gamma(5)}{12\cdot\Gamma(5-p)}t^{4-p}$$

$$\int_0^t (t-\tau)\left[{}_0D_\tau^p K(\tau)\right] + \lim_{\tau\to t^-} f(t-\tau)\left[{}_0D_\tau^{p-1}K(\tau)\right] =$$

$$= \int_0^t \frac{\Gamma(3)}{\Gamma(3-p)}t^{3-p}(t-\tau)d\tau + \lim_{\tau\to t^-} f(t-\tau)\frac{\Gamma(3)}{\Gamma(3-p)}t^{3-p} =$$

$$= \frac{\Gamma(3)}{\Gamma(3-p)}\left[t\frac{t^{3-p}}{3-p} - \frac{t^{4-p}}{4-p}\right] + 0 =$$

$$\Gamma(3)t^{4-p}\frac{4-p-(3-p)}{\Gamma(5-p)} = \frac{\Gamma(5)}{12\cdot\Gamma(5-p)}t^{4-p}$$

therefore the formula V.3

VI. Derivative Caputo & Laplace transformation

The Caputo derivative was proposed by Michel Caputo in 1967 to solve problems with initial values. If $m-1<p<m$ and the function f has a continuous derivative of order $m+1$ then the Caputo derivative is defined

$${}_\gamma^c D_t^p f(t) = \frac{1}{\Gamma(m-p)}\int_\gamma^t (t-\tau)^{m-p-1}f^{(m)}(\tau)d\tau \tag{VI.1}$$

The Caputo derivative transfers the m derivative of the integral directly to the function f ignoring the sum presented in Riemann Liouville formula

$${}_\gamma D_t^p f(t) = \frac{d^m}{dt^m}\frac{1}{\Gamma(m-p)}\int_\gamma^t (t-\tau)^{m-p-1}f(\tau)d\tau \tag{VI.2}$$

which for a fairly smooth function we have the expression

$${}_\gamma D_t^p f(t) = \sum_{k=0}^{m-1}\frac{f^{(k)}(\gamma)}{\Gamma(-p+k+1)}(t-\gamma)^{-p+k} + \frac{1}{\Gamma(m-p)}\int_\gamma^t (t-\tau)^{m-p-1}f^{(m)}(\tau)d\tau \tag{VI.3}$$

i.e. the Kaputo derivative completely ignores the sum in the type (VI.3)

An important difference between the Rimenann Liouville derivative and the Caputo derivative is the prices for the stable producer

From Caputo

$${}_\gamma D_t^p c = \frac{d^m}{dt^m}\frac{1}{\Gamma(1-p)}\int_\gamma^t (t-\tau)^{-p}\frac{dc}{d\tau}d\tau = 0,\ 0<p<1$$

But from Riemann - Liouville

$$ {}_{\gamma}D_t^p c = \frac{c}{\Gamma(1-p)}(t-\gamma)^{-p},\ 0<p<1 $$

Two values different from what we see.

The case for integration will be

$$ {}_0D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau = \frac{1}{\Gamma(p)} t^{p-1} \cdot f(t) $$

but

$$ \begin{aligned} & L\left\{\frac{t^{p-1}}{\Gamma(p)}\right\}(s) = \frac{1}{\Gamma(p)} \int_0^{\infty} e^{-st} t^{p-1} dt = \frac{1}{s^p \Gamma(p)} \int_0^{\infty} e^{-\tau} \tau^{p-1} d\tau = \\ & \frac{1}{s^p \Gamma(p)} \Gamma(p) = \frac{1}{s^p} \end{aligned} \tag{VI.4} $$

VII. Laplace transformation in fractional calculus

The Laplace transform of the function $f(t)$ is given by the following integral

$$ L\{f(t)\}(s) = \int_0^{+\infty} e^{-st} f(t) dt = F(s) \ \ \text{(VII.1) with } \operatorname{Re}(s) > s_0,\ s \in C,\ s_0 \in R $$

The Laplace transform is a continuous linear combination of oscillations of variable amplitude of the form

$$ e^{-st} = e^{-(\mathrm{Res})t}(\cos(\operatorname{Im} s)t - i \sin(\operatorname{Im} s)t) $$

The existence of the Laplace transformation requires f to be of exponential form

$$ a < s_0,\ M > 0,\ T > 0 : |f(t)| \le M e^{at},\ t > T $$

Also we have the form

$$ L\{f(t)\}(s) = s^n F(s) - \sum_{k=0}^{n-1} f^k(0) s^{n-k-1} = s^n F(s) - \sum_{k=0}^{n-1} f^{n-k-1}(0) s^k \tag{VII.2} $$

For the integration case, $p > 0$ applies

$$ {}_0D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau = \frac{1}{\Gamma(p)} t^{p-1} f(t) \tag{VII.3} $$

But also by Laplace

$$ L\left\{\frac{t^{p-1}}{\Gamma(p)}\right\}(s) = \frac{1}{\Gamma(p)} \int_0^{+\infty} e^{-st} t^{p-1} dt = \frac{1}{\Gamma(p)} \frac{1}{s^p} \int_0^{+\infty} e^{-\tau} \tau^{p-1} d\tau = \frac{1}{\Gamma(p)} \frac{1}{s^p} \Gamma(p) = s^{-p} \tag{VII.4} $$

From the two formulas above it follows

$$L\left\{{}_0D_t^{-p}f(t)\right\}(s) = s^{-p}F(s),\ p > 0 \tag{VII.5}$$

For the case of the fractional derivative if $-1 \le p < m,\ m = [p]+1$ then we introduce a function g with continuous derivatives up to order m and assume

$${}_0D_t^p f(t) = g^m(t)$$

$$g(t) = {}_0D_t^{-(m-p)}f(t) = \frac{1}{\Gamma(m-p)}\int_0^t (t-\tau)^{m-p-1}f(\tau)d\tau \tag{VII.6}$$

which implies that

$$G(s) = s^{-(m-p)}F(s),\ m-p > 0$$

$$L\left\{{}_0D_t^p f(t)\right\}(s) = L\left\{g^{(m)}(t)\right\}(s) = s^m G(s) - \sum_{k=0}^{m-1} s^k g^{(m-k-1)}(0) \tag{VII.7}$$

$$L\left\{{}_0D_t^p f(t)\right\}(s) = L\left\{{}_0D_t^{-(m-p)}g(t)\right\}(s) = s^{-(m-p)}G(s)$$

$$G(s) = s^{(m)}F(s) - \sum_{k=0}^{m-1} s^{m-k-1}f^k(0) = s^{(m)}F(s) - \sum_{k=0}^{m-1} s^k f^{(m-k-1)}(0) \tag{VII.8}$$

And final we have

$$L\left\{{}_0D_t^p f(t)\right\}(s) = L\left\{g^{(p)}(t)\right\}(s) = s^p F(s) - \sum_{k=0}^{m-1} s^{p-k-1}f^{(k)}(0) \tag{VII.9}$$

And final we have

$$L\left\{{}_0D_t^p f(t)\right\}(s) = L\left\{g^{(p)}(t)\right\}(s) = s^p F(s) - \sum_{k=0}^{m-1} s^{p-k-1}f^{(k)}(0) \tag{VII.9}$$

VIII. The function Mittag - Leffler

The well-known function with 2 parameters is

$$E_{a,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha\cdot n+\beta)}, \mathrm{Re}\, a > 0 \text{ and } \mathrm{Re}\,\beta > 0 \tag{VIII.1}$$

using the D'Alambert criterion

$$\frac{c_{n+1}}{c_n} = \frac{z}{\Gamma(\alpha)}B(\alpha n+b,a) = \frac{z}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha+1}\cdot\tau^{\alpha n+b-1}d\tau,\ \tau\in(0,1),\ \alpha > 0$$

Therefore

$$\lim \frac{c_{n+1}}{c_n} = \lim \tau^{\alpha n} = 0$$

a direct consequence of the definition are the following relations

$$E_{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = e^z$$

$$E_{1,2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+2)} = \sum_{n=0}^{\infty} \frac{1}{z} \frac{z^{n+1}}{(n+1)!} = \frac{e^z - 1}{z} \tag{VIII.2}$$

$$E_{1,m}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+m)} = \frac{1}{z^{m-1}} \sum_{n=0}^{\infty} \frac{z^{n+m-1}}{(n+m-1)!} =$$

$$\frac{1}{z^{m-1}} \left[\sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{k=0}^{m-2} \frac{z^k}{k!} \right] = \frac{1}{z^{m-1}} \left[e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!} \right]$$

$$E_{2,1}(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{\Gamma(2n+1)} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = \cosh z$$

$$\text{Also } E_{2,2}\left(z^2\right) = \sum_{n=0}^{\infty} \frac{z^{2n}}{\Gamma(2n+2)} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+)!} = \frac{\sinh z}{z} \tag{VIII.3}$$

The fractional sine and cosine functions in fractional calculus are given by the relations

$$Sc_a(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2-a)n+1}}{\Gamma((2-a)n+2)} = z E_{2-a,2}\left(-z^{2-a}\right), \; Sc_0(z) = \sin z$$

$$Cs_a(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2-a)n}}{\Gamma((2-a)n+1)} = E_{2-a,1}\left(-z^{2-a}\right), \; Cs_0(z) = \cos z \tag{VIII.4}$$

IX. The Laplace transformation and the Mittag-Leffler function

For the Mittag - Leffler function we have

$$\int_0^{+\infty} e^{-t} t^{\beta-1} E_{\alpha,\beta}\left(\pm z t^a\right) dt = \int_0^{+\infty} e^{-t} t^{\beta-1} \sum_{n=0}^{\infty} \frac{(\pm z)^n t^{an}}{\Gamma(\alpha v + \beta)} dt =$$

$$\sum_{n=0}^{\infty} \frac{(\pm z)^n t^{an}}{\Gamma(\alpha v + \beta)} \int_0^{+\infty} e^{-t} t^{\alpha n + \beta - 1} dt = \sum_{n=0}^{\infty} \frac{(\pm z)^n}{1} = \frac{1}{1 \mp z}, |z| < 1 \tag{IX.1}$$

and Therefore for $k = 0, 1, 2, 3 \ldots$ apply

$$\int_0^{+\infty} e^{-at} \left[t^{ak+\beta-1} E_{\alpha,\beta}^k \left(\pm \lambda t^a\right) \right] dt = \frac{k! s^{a-\beta}}{(s^\alpha \mp \lambda)^{k+1}}, \text{Res} > |\lambda|^{1/\alpha} \tag{IX.2}$$

Where k we denote the derivative k order with respect to the argument $\pm\lambda t^a$.

Therefore apply

$$L\left\{t^{ak+\beta-1}E_{\alpha,\beta}^{k}\left(\pm\lambda t^{a}\right)\right\}(s)=\frac{k!s^{a-\beta}}{\left(s^{\alpha}\mp\lambda\right)^{k+1}},\ \operatorname{Re}s>|\lambda|^{1/\alpha} \tag{IX.3}$$

For the Derivation of the function Mittag-Leffler we have

$$_{0}D_{t}^{\gamma}\left[t^{ak+\beta-1}E_{\alpha,\beta}^{k}\left(\pm\lambda t^{a}\right)\right]=t^{ak+\beta-\gamma-1}E_{\alpha,\beta-\gamma}^{k}\left(\lambda t^{a}\right),\ k\in Z^{+} \tag{IX.4}$$

Where ν in $R_{\iota}\beta>\gamma$ with derivative order ψ.

also the integration of the function Mittag-Leffler given at

$$_{0}D_{t}^{-\gamma}\left[t^{\beta-1}E_{\alpha,\beta}\left(\lambda t^{a}\right)\right]=t^{\beta+\gamma-1}E_{\alpha,\beta+\gamma}\left(\lambda t^{a}\right),\ \gamma>0 \tag{IX.5}$$

X. Equations of fractional analysis

If the function $f(t)$ is integrable in the interval $[0,t]$ and then $p_0=a_n+a_{n-1}+\ldots+a_1, 0<a_i<1, i=1,2,\ldots,n$ then the initial value problem

$$_{0}^{s}D_{t}^{p_n}y(t)=f(t), t\in[0,\tau] \tag{X.1}$$

where ${}_{0}^{s}D_{t}^{p_n}$ given from the ${}_{0}^{s}D_{t}^{p_k-1}y(t)\Big|_{t=0}=c_k,\ k=1,2,\ldots,n$

Where

$$_{0}^{s}D_{t}^{p_k-1}={}_{0}D_{t}^{a_k-1}\cdot{}_{0}D_{t}^{a_k-2}\ldots{}_{0}D_{t}^{a_2}\cdot{}_{0}D_{t}^{a_1},\ k=1,2,3\ldots,n$$

has a unique solution in the interval $[0,t]$ that given

$$y(t)=\frac{1}{\Gamma\left(p_n\right)}\int_0^{\tau}(t-\tau)^{p_n-1}f(\tau)d\tau+\sum_{i=1}^{n}\frac{c_i}{\Gamma\left(p_i\right)}t^{p_i-1} \tag*{(X.2)*}$$

Apply also the type

$$_{0}^{s}D_{t}^{p_n-1}\frac{t^{p_i-1}}{\Gamma\left(p_i\right)}=\left\{\begin{array}{cc}\dfrac{t^{p_i-p_k}}{\Gamma\left(p_i-p_k+1\right)}, & k<i\\ 1, & k=i\\ 0, & k>i\end{array}\right\} \tag*{(X.3)*}$$

Example.X1

Consider the sequential equation

${}_0^sD_t^{p_3}y(t) = t, t \in [0,\tau]$

$$\text{With} \quad \begin{aligned} p_3 &= a_1 + a_2 + a_3 \\ p_2 &= a_1 + a_2 \\ p_1 &= a_1 \end{aligned} \qquad \text{and} \quad a_3 = \frac{1}{2}, a_2 = \frac{1}{3}, a_1 = \frac{1}{2}.$$

$$y(t) \text{ must satisfy both the initial conditions} \quad \begin{aligned} {}_0^sD_t^{p_1-1}y(t)\Big|_0 &= 1 \\ {}_0^sp_t^{p_2-1}y(t)\Big|_0 &= 2 \\ {}_0^sp_t^{p_3-1}y(t)\Big|_0 &= -1 \end{aligned}$$

Solution

From (X.2) we take $p_3 = \frac{1}{2} + \frac{1}{3} + \frac{1}{2} = 1 + \frac{1}{3} = \frac{4}{3}$ and therefore we have

$$\begin{aligned} y(t) =& \frac{1}{\Gamma(p_n)} \int_0^t (t-\tau)^{p_n-1} f(\tau) d\tau + \sum_{i=1}^{n} \frac{c_i}{\Gamma(p_i)} t^{p_i-1} = \\ & \frac{1}{\Gamma\left(\frac{4}{3}\right)} \int_0^t (t-\tau)^{4/3} \tau d\tau + \frac{1}{\Gamma\left(\frac{1}{2}\right)} t^{1/2-1} + \frac{2}{\Gamma\left(\frac{5}{6}\right)} t^{5/6-1} + \frac{-1}{\Gamma\left(\frac{4}{3}\right)} t^{4/3-1} = \\ & \frac{1}{\Gamma\left(\frac{4}{3}\right)} t^{7/3} B\left(\frac{4}{3}, 2\right) + \frac{1}{\sqrt{\pi}} t^{-1/2} - \frac{12}{\Gamma(-1/6)} t^{-1/6} - \frac{3}{\Gamma(1/3)} t^{1/3} = \\ & \frac{1}{\Gamma\left(\frac{10}{3}\right)} t^{7/3} + \frac{1}{\sqrt{\pi}} t^{-1/2} - \frac{12}{\Gamma(-1/6)} t^{-1/6} - \frac{3}{\Gamma(1/3)} t^{1/3} \end{aligned}$$

XI. Methods of solving linear equations

XI.1 Solving the equation

$$a_o D_t^p y(t) = f(t),\ t > 0,\ a = \text{ const},\ \ p > 0,\ m - 1 = [p]$$

using the Laplace transformation will be have

$$aL\left\{{}_0D_t^p y(t)\right\}(s) = L\{f(t)\}(s) \text{ or}$$

$$as^p L\{y(t)\}(s) - a\sum_{k=0}^{m-1} s^k \left({}_0D_t^{p-k-1}y(t)\right)|_{t=0} = L\{f(t)\}(s)$$

$m-1=[p]$ therefore

$$L\{y(t)\}(s) = \frac{1}{s^p}\sum_{k=0}^{m-1} s^k \left({}_0D_t^{p-k-1}y(t)\right)|_{t=0} + \frac{1}{as^p}L\{f(t)\}(s)$$

But $L\left\{\dfrac{t^{p-1}}{\Gamma(p)}\right\}(s) = s^{-p}$ then

$$L\{y(t)\}(s) = \sum_{k=0}^{m-1}\left({}_0D_t^{p-k-1}y(t)\right)\Big|_{t=0} L\left\{\frac{t^{p-k-1}}{\Gamma(p-k)}\right\}(s) + \frac{1}{a}L\left\{\frac{t^{p-1}}{\Gamma(p)}\right\}(s)\cdot L\{f(t)\}(s) =$$

$$= L\left\{\sum_{k=0}^{m-1}\left({}_0D_t^{p-k-1}y(t)\right)\Big|_{t=0}\frac{t^{p-k-1}}{\Gamma(p-k)}\right\}(s)\right\} + \frac{1}{a}L\left\{\frac{t^{p-1}}{\Gamma(p)}\right\}(s)\cdot f(t)\right\}(s)$$

The final solution is

$$y(t) = \sum_{k=0}^{m-1}\left({}_0D_t^{p-k-1}y(t)\right)\Big|_{t=0}\frac{t^{p-k-1}}{\Gamma(p-k)} + \frac{1}{a\cdot\Gamma(p)}\int_0^t (t-\tau)^{p-1}f(\tau)d\tau =$$

$$= \sum_{k=0}^{m-1}\left({}_0D_t^{p-k-1}y(t)\right)\Big|_{t=0}\frac{t^{p-k-1}}{\Gamma(p-k)} + \frac{1}{\alpha}{}_0D_t^{-p}f(t)$$

If the initial conditions are zero then the following applies

$$y(t) = \frac{1}{\alpha}{}_0D_t^{-p}f(t)$$

XI.2 Methods of solving linear equations by 2 terms

We consider the equation

$$a{}_0D_t^p y(t) + by(t) = f(t),\ t>0,\ a,\ b = const,\ p>0,\ m-1=[p]$$

we take the laplace transformation

$$aL\left\{{}_0D_t^p y(t) + by(t)\right\}(s) = L\{f(t)\}(s) \text{ or}$$

$$as^p L\{y(t)\}(s) - a\sum_{k=0}^{m-1} s^k\left({}_0D_t^{p-k-1}y(t)\right)|_{t=0} + L\{by(t)\}(s) = L\{f(t)\}(s)$$

$m-1=[p]$ therefore

$$as^pY(s)=a\sum_{k=0}^{m-1}s^k\left({}_0D_t^{p-k-1}y(t)\right)\Big|_{t=0}+bY(s)=F(s)\text{ or}$$

$$Y(s)=\frac{F(s)}{as^p+b}+a\sum_{k=0}^{m-1}\frac{s^k}{as^p+b}\left({}_0D_t^{p-k-1}y(t)\right)\Big|_{t=0} \tag{XI.2.1}$$

From (IX.2) we have for $k=0$

$$L\left\{t^{\gamma-1}E_{\delta,\gamma}^{k}\left(\pm\lambda t^{\delta}\right)\right\}(s)=\frac{k!s^{\delta-\gamma}}{\left(s^{\delta}\mp\lambda\right)^{k+1}},\mathrm{Res}>|\lambda|^{1/\alpha}\Leftrightarrow$$

$$L\left\{t^{\gamma-1}E_{\delta,\gamma}\left(\pm\lambda t^{\delta}\right)\right\}(s)=\frac{s^{\delta-\gamma}}{\left(s^{\delta}\mp\lambda\right)},\mathrm{Res}>|\lambda|^{1/\alpha}$$

The (XI.2.1) becomes

$$L\{y(t)\}(s)=\frac{L\{f(t)\}(s)}{as^p+b}+\frac{1}{a}\sum_{k=0}^{m-1}\frac{s^k}{s^p+\frac{b}{a}}\left({}_0D_t^{p-k-1}y(t)\right)\Big|_{t=0}=$$

$$\frac{1}{a}L\left\{t^{p-1}E_{p,p}\left(-\frac{b}{a}t^p\right)\right\}(s)\cdot L(f(t)\}(s)+\frac{1}{a}\sum_{k=0}^{m-1}\left({}_0D_t^{p-k-1}y(t)\right)\Big|_{t=0}L\left\{t^{p-k-1}E_{p,p-k}\left(-\frac{b}{a}t^p\right)\right\}(s)=$$

$$\frac{1}{a}L\left\{t^{p-1}E_{p,p}\left(-\frac{b}{a}t^p\right)\cdot f(t)\right\}(s)+L\left\{\frac{1}{a}\sum_{k=0}^{m-1}\left({}_0D_t^{p-k-1}y(t)\right)\Big|_{t=0}t^{p-k-1}E_{p,p-k}\left(-\frac{b}{a}t^p\right)\right\}(s)\Leftrightarrow$$

$$y(t)=\frac{1}{a}\int_0^t(t-\tau)^{p-1}E_{p,p}\left(-\frac{b}{a}(t-\tau)^p\right)\cdot f(\tau)d\tau+\frac{1}{a}\sum_{k=0}^{m-1}\left({}_0D_t^{p-k-1}f(t)\right)\Big|_{t=0}t^{p-k-1}E_{p,p-k}\left(-\frac{b}{a}t^p\right) \tag{XI.2.2}$$

XI.3 Methods of solving linear equations by 3 terms

Consider the equation

$$a_0D_t^py(t)+b_0D_t^qy(t)+cy(t)=f(t),\ t>0,\ a,b,c=\text{ const}$$

$$p>0,\ q>0,\ m-1=[p],\ n-1=[q].$$

we also assume that all terms at $t=0$ are zero. The Laplace transformation of the original takes the form

$m-1=[p],\ n-1=[q]$ therefore

$as^pY(s)+bs^qY(s)+cY(s)=F(s)$ or

$$Y(s)=\frac{F(s)}{as^p+bs^q+c} \tag{XI.3.1}$$

But apply

$$\frac{Y(s)}{F(s)} = \frac{1}{c}\frac{cs^{-q}}{as^{p-q}+b} \cdot \frac{1}{1+\dfrac{cs^{-q}}{as^{p-q}+b}} =$$

$$\frac{1}{c}\sum_{k=0}^{\infty}(-1)^k\left(\frac{c}{a}\right)^{k+1}\frac{s^{-kq-q}}{\left(s^{p-q}+\frac{b}{a}\right)^{k+1}} \tag{XI.3.2}$$

we use the type

$$L\left\{t^{\delta k+\gamma-1}E^k_{\delta,\gamma}\left(\pm\lambda t^{\delta}\right)\right\}(s) = \frac{k!s^{\delta-\gamma}}{\left(s^{\delta}\mp\lambda\right)^{k+1}}, \operatorname{Re} s > |\lambda|^{1/\delta} \text{ then}$$

$$Y(s) = L\left\{\frac{1}{c}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{c}{a}\right)^{k+1}t^{(k+1)p-1}E^k_{p-q,p+kq}\left(\frac{-b}{a}t^{p-q}\right)\right\}F(s) =$$

$$L\left\{f(t)\cdot\frac{1}{a}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{c}{a}\right)^{k}t^{(k+1)p-1}E^k_{p-q,p+kq}\left(\frac{-b}{a}t^{p-q}\right)\right. \Leftrightarrow$$

$$y(t) = \frac{1}{a}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{c}{a}\right)^{k}\int_0^{\infty}f(t-\tau)\cdot\tau^{(k+1)p-1}E^k_{p-q,p+kq}\left(\frac{-b}{a}\tau^{p-q}\right)d\tau \tag{XI.3.3}$$

of course the expansion in the geometric series we used requires the constraint

$$\left|\frac{cs^{-q}}{as^{p-q}+b}\right| = \left|\frac{c}{as^p+bs^q}\right| < 1$$

We have followed a difficult method and in the future we will see more.

XI.4 Methods of solving homogeneous linear equations by n order

XI.4.I. We consider the linear and homogeneous fractional differential equation with constant coefficients and order of derivatives

$$r_n, r_{n-1}, \ldots r_0$$

$$\begin{aligned} &D_t^{r_m}y(t)+b_1D_t^{r_{m-1}}y(t)+\ldots+b_mD_t^{r_0}\big]\,y(t)=0,\ b_i = \text{ const},\ \ i=0,1,\ldots,n \\ &0\le r_0<r_1<\ldots<r_{n-1}<r_n \end{aligned} \tag{XI.4.1}$$

We write all the explicit real numbers in their reduced form and calculate the least common multiple q, of the denominators of all non-zero explicit numbers $r_i, i=0,1,2,\ldots n$ we define the inverse of $q, n = r_n\cdot q, v=\frac{1}{q}$ the equation is done at

$$D_t^{n\cdot v}+b_1D_t^{(n-1)v}+\ldots+b_{n-1}D^v+b_nD_t^0\big]\,y(t)=0, b_i = \text{ const },\ i=0,1,2,\ldots n$$

We therefore define **(XI.4.1)** as a linear homogeneous fractional differential equation of order (n, q) with constant coefficients.

XI.4.II.Theorem 1.

We consider the linear and homogeneous fractional differential equation with constant coefficients and order of derivatives

$$\begin{aligned} & \left.D_t^{n \cdot v}+b_1 D_t^{(n-1) \cdot v}+\ldots+b_{n-1} D^v+b_n D_t^0\right] y(t)=0, \\ & b_i= \text{ const } , i=0,1,2, \ldots n \end{aligned} \tag{XI.4.2}$$

Where $\nu=\frac{1}{q}$ and $D^p={ }_0 D_t^p$ for each p and the corresponding polynomial characteristic

$$P(x)=x^n+a_1 x^{n-1}+a_2 x^{n-2}+\ldots+a_n \text{ consider the function } y_1(t)=L^{-1}\left\{\frac{1}{P\left(s^p\right)}\right\}(t)$$

If apply $v \cdot n \leq N$ and N is the oddest integer then the functions $y_1(t), y_2(t), \ldots, y_N(t)$ defined by the relations $y_{j+1}(t)=D y_j(t)=D^J y_1(t)$ for $j=1,2, \ldots, N-1$ are the linear independent solutions of the equation (XI.4.2).

Proof.

The equation (XI.4.2) is written

$$P\left(D^v\right) y(t)=0$$

and using the type

$$L\left\{D^p y(t)\right\}(s)=s^p Y(s)-\sum_{k=0}^{m-1}\left.\left(D^{p-1-k} y(t)\right)\right|_{t=0} s^k, m-1<p<m \text{ and we get}$$

$$L\left\{P\left(D^v\right) y(t)\right\}(s)=L\left\{\left.\left(D^v\right)^n+\sum_{\sigma=1}^n a_\sigma\left(D^v\right)^{n-\sigma}\right] y(t)\right\}(s)=$$

$$P\left(s^v\right) Y(s)-\sum_{\sigma=1}^{[n v]}\left.\left(D^{v n-1-k} y(t)\right)\right|_{t=0} s^k-\sum_{\sigma=1}^{n-1} a_\sigma \sum_{k=0}^{[v(n-\sigma)]}\left.\left(D^{(v-\sigma \mid v-1-k} y(t)\right)\right|_{t=0} s^k \tag{XI.4.3}$$

$$L\left\{P\left(D^v\right) y(t)\right\}(s)=P\left(s^v\right) Y(s)-\sum_{k=0}^{N-1} B_k(y) s^k \text{ and final}$$

$$y(t)=L^{-1}\left\{\frac{\sum_{\{=0}^{N-1}\left(B_k(y) s^k\right.}{P\left(s^v\right)}\right\}(t) \tag{XI.4.4}$$

Checking the functions

$$y_1(t), y_1^{(1)}(t), \ldots, y_1^{(n)}(t) \text{ to be linearly independent}$$

Apply

$$L\left\{y_1^{(N)}\right\} = L\left\{D^{N-1} y_1\right\}$$

$$\begin{aligned} &\text{if } N = [nv] \text{ then } y_N(0) = y_1^{N-1}(0) = 1 \\ &\text{if } N > nv \text{ then} \\ &y_N(0) = y_1^{N-1}(0) = +\infty \end{aligned}$$

we consider the linear combination

$$\sum_{k=1}^{N-1} \lambda_k y_k(t) + \lambda_N y_N(t) = 0 \text{ and for } t = 0$$

$$\sum_{k=1}^{N-1} \lambda_k y_k(t) + \lambda_N y_N(t) = 0$$

Because apply

$$y_N(0) \neq 0 \Rightarrow \lambda_N = 0$$

Therefore will be linearly independent

Part.III Approach solutions of fractional differential equations

XII.1 Approximate algorithm of solution simple fractional differential equations

Initial value problem (i.v.p.), while a fractional differential equation, equipped with boundary conditions (i.e. conditions determined by values of the unknown function and its derivatives in more than one points) constitutes a boundary value problem (b.v.p.). Functions satisfying the fractional differential equation and the initial or boundary conditions, respectively, of these problems. The generalised i.v.p. (i.e. the i.v.p. for a differential equation containing more than one derivative of the unknown function) for linear ones, fractional differential equations of order $\sigma_n > 0 \, \forall \, n \in N_{>1}$ so the generalized form of the fractional differential equation

$$\sum_{j=0}^{n} a_{jx} D_{0^+}^{\sigma_{n-j}} y(x) = f(x),\ \sigma_0 = 0,\ n \in N,\ \sigma_n \in R_{>0} \tag{XII.1.1}$$

with initial value

$$D_{0^+}^{\sigma_k} y\left(0^+\right) = b_k,\ k = 1, 2, \ldots, n \in N,\ b_k \in R \tag{XII.1.2}$$

where $f \in L_1(0, T)$ for some $T > 0$ and $f(x) = 0$ for all $x > T$, is solved using Laplace transforms and it is shown that the solution of $y \in L_1(0, T)$ is unique.

The approximate solution is mainly in the form of Grunwald-Letnikov fractional derivative is closely related to the Caputo and RiemannLiouville derivatives

According to the Grunwald-Letnikov Theory respectively, we define the derivative of order p

$$ {}_{\gamma}D_x^p f(t) = \lim_{\substack{h\to 0 \\ nh=t-\gamma}} \frac{1}{h^p}\sum_{r=0}^{n}(-1)^r \begin{pmatrix} p \\ r \end{pmatrix} f(t-r\cdot h),\ p>0 $$

$$ \frac{d^m}{dt^m}\frac{1}{\Gamma(m-p)}\int_{\gamma}^{t}(t-\tau)^{m-p-1}f(\tau)d\tau $$

The relation $\sum_{j=0}^{n} a_j D_{0^+}^{\sigma_{n-j}} y(x) = f(x),\ \sigma_0 = 0,\ n \in N,\ \sigma_n \in R_{>0}$ we can transform it into a form functional algorithm as

$$ \sum_{j=0}^{n} a_{jx} D_{0^+}^{\sigma_{n-j}} y(x) = f(x),\ \sigma_0 = 0,\ n \in N,\ \sigma_n \in R_{>0} \Rightarrow $$

$$ \left\langle \begin{array}{l} \sum_{j=1}^{n} a_j \left(\frac{1}{h^{p_j}} \sum_{k=0}^{x_i} (-1)^k \begin{pmatrix} p_j \\ \\ k \end{pmatrix} \right) y\left(x_i - k\right) + a_0 y\left(x_i\right) = f\left(x_t\right) = f\left(x_i \cdot h\right) \\ \\ p_0 = 0,\ x_t = x_i \cdot h,\ 1 \le x_i \le \frac{x_t}{h},\ h \in (0,1),\ \{p_j, a_j\} \in R^+ \end{array} \right\rangle \tag{XII.1.3} $$

But apply

$$ (-1)^k \begin{pmatrix} p_j \\ k \end{pmatrix} = \frac{\Gamma\left(\kappa - p_j\right)}{\Gamma\left(-p_j\right)\cdot\Gamma(\kappa+1)} \text{ XII.1.4 we get the final form} $$

$$ \left\langle \begin{array}{l} \sum_{j=1}^{n} a_j \left(\frac{1}{h^{p_j}} \sum_{k=0}^{x_i} \frac{\Gamma\left(k - p_j\right)}{\Gamma\left(p_j\right)\cdot\Gamma(k+1)} y\left(x_i - k\right) + a_0 y\left(x_i\right) = f\left(x_t\right) = f\left(x_i \cdot h\right) \right. \\ p_0 = 0, x_t = x_i \cdot h, 1 \le x_i \le \frac{x_t}{h}, h \in (0,1), \{p_j, a_j\} \in R^+ \end{array} \right\rangle \tag{XII.1.5} $$

If excluding the case with $k = 0$ in the Sum we get the final form

$$ \left\langle \begin{array}{l} y\left(x_i\right) = \frac{1}{\sum_{j=0}^{n} \frac{a_j}{h^{p_j}}} \left(f\left(x_i \cdot h\right) - \sum_{j=1}^{n} a_j \left(\frac{1}{h^{p_j}} \sum_{k=1}^{x_i} \frac{\Gamma\left(k - p_j\right)}{\Gamma\left(p_j\right)\cdot\Gamma(k+1)} y\left(x_i - k\right) \right) \right. \\ p_0 = 0,\ x_t = x_i \cdot h,\ 1 \le x_i \le \frac{x_t}{h},\ h \in (0,1),\ \{p_j, a_j\} \in R^+ \end{array} \right\rangle \tag{XII.1.6} $$

XII.2 Approximate algorithm of solution transcendental fractional differential equations

Starting from the relation

$$\left\langle \begin{array}{l} \displaystyle\sum_{j=1}^{n} a_j \left(\frac{1}{h^{p_j}} \sum_{k=0}^{x_i} \frac{\Gamma(k-p_j)}{\Gamma(p_j)\cdot\Gamma(k+1)} y(x_i-k) + a_0 y(x_i) = f(x_t) = f(x_i\cdot h) \right. \\ p_0=0,\ x_t = x_i\cdot h,\ 1\le x_i \le \frac{x_t}{h},\ h\in(0,1),\ \{p_j,a_j\}\in R^+,\ j\in N \end{array} \right\rangle \tag{XII.2.1}$$

assuming we have a generalized form for the simple $y(x_i)$ we will havethe form $G(y(x_i),\ l\cdot x_t),\ l\in R_{\neq 0}$ which is a more general function, we will transform the above

$$\left\langle \begin{array}{l} \displaystyle\sum_{j=1}^{n} a_j \left(\frac{1}{h^{p_j}} \sum_{k=0}^{x_i} \frac{\Gamma(k-p_j)}{\Gamma(p_j)\cdot\Gamma(k+1)} y(x_i-k) + a_0 G(y(x_i), l\cdot x_t) = f(x_t) = f(x_i\cdot h) \right. \\ p_0=0,\ x_t = x_i\cdot h,\ 1\le x_i \le \frac{x_t}{h},\ h\in(0,1),\ \{p_j,a_j\}\in R^+,\ j\in N,\ l\in R_{\neq 0} \end{array} \right\rangle \Leftrightarrow$$

form $\left\langle \begin{array}{l} \displaystyle a_0 G(y(x_i), l\cdot x_t) = f(x_i\cdot h) - \sum_{j=1}^{n} a_j \left(\frac{1}{h^{p_j}} \sum_{k=0}^{x_i} \frac{\Gamma(k-p_j)}{\Gamma(p_j)\cdot\Gamma(k+1)} y(x_i-k) \right. \\ p_0=0, x_t = x_i\cdot h, 1\le x_i \le \frac{x_t}{h}, h\in(0,1), \{p_j,a_j\}\in R^+, j\in N, l\in R_{\neq 0} \end{array} \right\rangle$ and we take the

final form

$$\left\langle \begin{array}{l} \displaystyle a_0 G(y(x_i), l\cdot x_t) - \sum_{j=1}^{n} \frac{a_j}{h^{p_j}} y(x_i) = f(x_i\cdot h) - \sum_{j=1}^{n} a_j \left(\frac{1}{h^{p_j}} \sum_{k=1}^{x_i} \frac{\Gamma(k-p_j)}{\Gamma(p_j)\cdot\Gamma(k+1)} y(x_i-k) \right. \\ p_0=0,\ x_t = x_i\cdot h,\ 1\le x_i \le \frac{x_t}{h},\ h\in(0,1),\ \{p_j,a_j\}\in R^+,\ j\in N,\ l\in R_{\neq 0} \end{array} \right\rangle \tag{XII.2.2}$$

This is the transcendental differential fractional equation of general form which, in order to solve it, we need to solve a transcendental equation of the form

$$a_0 G(y(x_i), l\cdot x_t) - \sum_{j=1}^{n} \frac{a_j}{h^{p_j}} y(x_i) = 0 \tag{XII.2.3}$$

as $y(x_i)$ every time we look for the next approximate solution to x_i.

XII.3 Program for finding of $y[xi]$ for a fractional transcendental differential equation in mathematica language, method (GRIM):

Off[FindRoot::bddir,FindRoot::jsing,FindRoot::precw,Power::infy,Infinity::indet,General::cvmit,General: :stop, ReplaceAll::reps,Divide::infy,FindRoot:1stol,General::munfl, General::stop, General::precw];

'program data'

$n =$ Input [Number fractional equations $=$ "];

$p_j =$ Input ["Values of $p_j =$ "]

$a_j =$ Input ["Value of $a_j =$ "]

$l =$ Input ["Value of coefficient $t =$ "]

$q =$ Input ["Value of approch $h = 1/q =$ "]

$u =$ Input ["Value of t to search for, $u =$ "]

$c =$ Input ["Initial value for find Root $z =$ "]

$y[0] =$ Input ["Initial value for $y[0] =$ "]

'program execution'

$$G\left[y_{-}, x_{-}\right] := g(y, 1 \cdot x) + N\left[\sum_{j=1}^{n}\left(\frac{\alpha_j}{h^{p_j}}\right) \cdot y;\right]$$

$h := 1/q; w[i] := 0; S[x \quad T := s[x]$

$F\left[r_{\_}\right] := f(r);$

$A[1] := y[0]$

'Loop'

For$[i = 1, i <= q, i++,$

$s = 0; t = i/q$

$j = j + 1$

$y[i-1] = A[j]$

$$w[i] = N\left[F[t] - \sum_{j=1}^{n}\left(\frac{\alpha_j}{h^{p_j}} \sum_{k=1}^{i} \frac{\Gamma\left(k - p_j\right)}{\Gamma\left(p_j\right) \times \Gamma(k+1)} y(i-k)\right)\right]$$

$s = N\lceil z]/.$FindRoot $[G[z, t] - w[i] == 0, \{z, c\},$ WorkingPrecision- $> 10]$

$y[i] = s$

$A[j+1] = N[y[i], 10]$

If $[t >= u \&\& t < u + 1/q$

Print["y[",N[t],"] $=$ ", $N[y[i], 10]$, ", Error $=$ ", $Abs[y[i]-g[t]]$], Loopback]]

ListPlot[Table[A[i],{i,1,1000,1}], PlotRange $\rightarrow \{0, 1.1\}$, DataRange $\rightarrow \{0, 1.1\}$, AxesStyle $\rightarrow$

Arrowheads [$\{0, 0.03\}$], AxesLabel $\rightarrow \{"x", "y(x)"\}$, Axes $\rightarrow$ True

In this program we assume that we solve the Fractional differential equation of transcendental form

$$\left\langle \begin{array}{l} a_0 G\left(y\left(x_i\right), l \cdot x_t\right) - \sum_{j=1}^{n} \frac{a_j}{h^{p_j}} y\left(x_i\right) = f\left(x_i \cdot h\right) - \sum_{j=1}^{n} a_j \left(\frac{1}{h^{p_j}} \sum_{k=1}^{x_i} \frac{\Gamma\left(k-p_j\right)}{\Gamma\left(p_j\right) \cdot \Gamma(k+1)} y\left(x_i - k\right) \right. \\ p_0 = 0,\ x_t = x_i \cdot h,\ 1 \le x_i \le \frac{x_t}{h},\ h \in (0,1),\ \{p_j, a_j\} \in R^+,\ j \in N,\ l \in R_{\neq 0} \end{array} \right\rangle$$

and we assume that the correct theoretical function for y is $s(x)$ and from this we calculate the error with the current calculated value $y(i) - s(t)$. There is a correspondence in the variables where xi is i and xt is t in the program. The value for u is the value of xi for which we want to calculate the value of $y(xi)$ i.e in our program are respectively i and $y(i)$. the most interesting thing is that $i \neq t$ in the whole program process otherwise we would get incorrect results.

XIV. Numerical Examples for a fractional simple and transcendental differential equation in mathematica language

In this section, we present computational results of some examples to support our theoretical results. Let us consider an initial value problem for one of the simplest differential equations of fractional order appearing in applied problems:

$$\left\langle \begin{array}{l} \sum_{j=1}^{n} a_j \left(\frac{1}{h^{p_j}} \sum_{k=0}^{x_i} \frac{\Gamma\left(k-p_j\right)}{\Gamma\left(p_j\right) \cdot \Gamma(k+1)} y\left(x_i - k\right) + a_0 y\left(x_i\right) = f\left(x_t\right) = f\left(x_i \cdot h\right) \right. \\ p_0 = 0,\ x_t = x_i \cdot h,\ 1 \le x_i \le \frac{x_t}{h},\ h \in (0,1),\ y^{(k)}(0) = 0,\ k \in N_{\le n-1},\ \{p_j, a_j\} \in R^+ \end{array} \right\rangle \tag{XIV.1}$$

Example 1. As the first example, we consider the following initial value problem:

$$a_1 \sum_{k=1}^{1} \frac{1}{h^{p_1}} \sum_{k=0}^{x_i} \frac{\Gamma\left(k-p_1\right)}{\Gamma\left(p_1\right) \cdot \Gamma(k+1)} y\left(x_i - k\right) + y\left(x_i\right) = x_t^4 + \frac{24 x_t^{4-a}}{\Gamma(5-\alpha)}, \left(x_t = x_i \cdot h\right), y(0) = 0$$

for $p_1 = \frac{1}{16}, \frac{1}{8}, \frac{1}{2}, 1$

The results show the efficiency of the approximate solution in $y(0.6)$ for $q = 1000, h = 1/q$ and the relevant graph. The solution is done using the general program XII.3, with

$$f\left(x_t\right) = x_t^4 + \frac{24 x_t^{4-p_1}}{\Gamma\left(5-p_1\right)}, \left(x_t = x_i \cdot h\right), a_1 = 1$$

Results

1. $p_1 = \frac{1}{16},\ y(0.6) = 0.129263$
2. $p_1 = \frac{1}{8},\ y(0.6) = 0.128956$
3. $p_1 = \frac{1}{2},\ y(0.6) = 0.129753$
4. $p_1 = 1,\ y(0.6) = 0.129974$

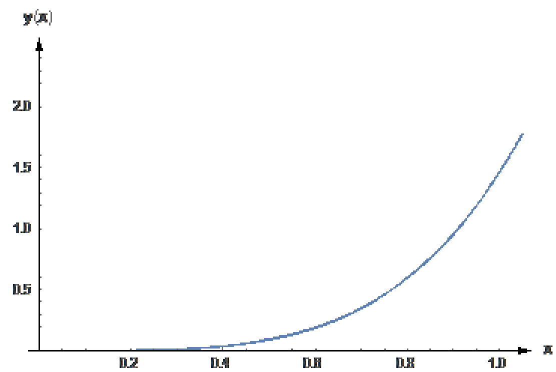

Figure 2:

Example 2. As a second example, consider the nonlinear ordinary transcendental differential equation

$$a_1 \sum_{k=1}^{1} \frac{1}{h^{p_1}} \sum_{k=0}^{x_i} \frac{\Gamma(k-p_1)}{\Gamma(p_1)\cdot\Gamma(k+1)} y(x_i - k) + (y(x_i))^2 = \frac{120x_t^{5-p_1}}{\Gamma(6-p_1)} + \frac{72x_t^{4-p_1}}{\Gamma(5-p_1)} + \frac{12x_t^{3-p_1}}{\Gamma(4-p_1)} + k\left(x_t^5 - 3x_t^4 + 2x_t^3\right)^2$$

$a_1 = 1, (x_t = x_i \cdot h, y(0) = 0), k = 1$

for $p_1 = \frac{1}{2}$

The exact solution is a polynomial of degree five

$$y(x_t) = k\left(x_t^5 - 3x_t^4 + 2x_t^3\right)^2$$

With this method we use the approximate solution to $y(0.6)$ for $N = 1000, h = 1/N$ we obtain the error and the associated graph. We use the program for the transcendental differential equation. The solution is done using the program XII.3.

Results

$$p_1 = \frac{1}{2}, y(0.6) = 0.1210607325,\ \text{error } = 0.00010073$$

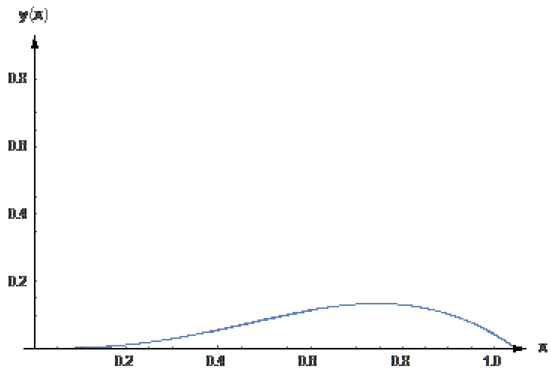

Figure 3:

Example 3. As our last example, we consider the following fractional nonlinear transcendental differential equation borrowed from

$$a_1 \sum_{k=1}^{1} \frac{1}{h^{p_1}} \sum_{k=0}^{x_i} \frac{\Gamma(k-p_1)}{\Gamma(p_1)\cdot\Gamma(k+1)} y(x_i-k) + \left(x_t - \frac{1}{2}\right) \cdot \mathrm{Sin}\left(y\left(x_i\right)\right) = 0.8x_t^3, a_1 = 1, p_1 = 0.28$$

subject to a nonzero initial condition, we show the numerical solutions for $N = 1000$, for several values of initial condition $y(0) = 1.2, 1.3, 1.4, 1.5$.

With this method we use the approximate solution to $y(0.6)$ for $N = 1000, h = 1/N$ for different initials values, we obtain the associated graph. We use the program for the transcendental differential equation.

Results

1. $p_1 = 0.28, y(0) = 1.2, y(0.6) = 0.1057900639$
2. $p_1 = 0.28, y(0) = 1.3, y(0.6) = 0.1062224499$
3. $p_1 = 0.28, y(0) = 1.4, y(0.6) = 0.1066546259$
4. $p_1 = 0.28, y(0) = 1.5, y(0.6) = 0.1070865772$

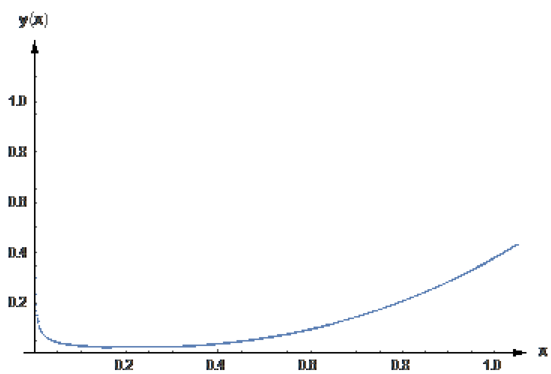

Figure 4:

We use program XII.3 and according to it we solve the transcendental equation $\mathrm{Sin}(z)\cdot\left(x_t-\frac{1}{2}\right)+z\cdot\frac{a_1}{h^{p1}}=0,\ x_t=\frac{i}{q},\ q=1000,\ 1\le i\le q$ which in this case is solved by mathematica-12 command and Newton's method. In general, however, we can solve it with the GRIM method and find infinite solutions.

XV.1 Approximate algorithm of solution transcendental fractional differentials equations of systems

We consider the generalized transcendental fractional differentials equations of system

$$
\left|\begin{array}{l}
g_1\left(x_t,y^1_{x_i},\dots,y^n_{x_i}\right)\sum_{j=1}^{q_1}a^1_j\frac{1}{h^{p^1_j}}\sum_{k=0}^{x_i}\frac{\Gamma\left(k-p^1_j\right)}{\Gamma\left(p^1_j\right)\cdot\Gamma(k+1)}y^1\left(x_i-k\right)+a^1_0y^1\left(x_i\right)\cdot w_1\left(x_t,y^1_{x_i},\dots,y^n_{x_i}\right)=\\
\qquad=f_1\left(x_t,y^1_{x_i},\dots y^n_{x_i}\right)\\
g_2\left(x_t,y^1_{x_i},\dots,y^n_{x_i}\right)\sum_{j=1}^{q_2}a^2_j\frac{1}{h^{p^2_j}}\sum_{k=0}^{x_i}\frac{\Gamma\left(k-p^2_j\right)}{\Gamma\left(p^2_j\right)\cdot\Gamma(k+1)}y^2\left(x_i-k\right)+a^2_0y^2\left(x_i\right)\cdot w_2\left(x_t,y^1_{x_i},\dots,y^n_{x_i}\right)=\\
\qquad=f_2\left(x_t,y^1_{x_i},\dots y^n_{x_i}\right)\\
\qquad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\qquad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
g_n\left(x_t,y^1_{x_i},\dots,y^n_{x_i}\right)\sum_{j=1}^{q_n}a^n_j\frac{1}{h^{p^n_j}}\sum_{k=0}^{x_i}\frac{\Gamma\left(k-p^n_j\right)}{\Gamma\left(p^n_j\right)\cdot\Gamma(k+1)}y^n\left(x_i-k\right)+a^n_0y^n\left(x_i\right)\cdot w_n\left(x_t,y^1_{x_i},\dots,y^n_{x_i}\right)=\\
\qquad=f_n\left(x_t,y^1_{x_i},\dots y^n_{x_i}\right)\\
p_0=0,\ b_0,\ x_t=x_i\cdot h,\ 1\le x_i\le\frac{x_t}{h},\ h\in(0,1),\ \left\{a^i_j,p^i_j,q_j\right\}\in R^+,\ y^{(l)}(0)=\\
\qquad=b_l,\ \left\{k\in N_{\le n_i-1}\right\},n,i,j,b_i,l\in N^+
\end{array}\right| \tag{XV.1.1}
$$

So we have the generalized system with fractional differentials equations and multiple n-number and we will try according to the previous theory to solve such a system. Such a program requires a strong functional programming language because it will have many iterations. Mathematica-12 is a powerful tool in C language, and it gives good approximations. In the general case we use the GRIM method or the simple Newton method, if it is generally about transcendental equations within the fractional differentials, with corresponding function.

XV.2 Program for finding of $y^l\left(x_i\right),\ y^{(l)}(0)=b_l,\ l\in N^+$ for a fractional transcendental differentials equations of system, in mathematica language, method (GRIM-Newton):

Here we give a 2 dimensional program with 2 variables $y^l\left(x_i\right),\ 1\le l\le 2,\ y^{(l)}(0)=b_l$, with 2 fractional differentials equations and generally a transcendental system and ask for its solution.

Example 1. In the given case we solve the system

$$\left\langle \begin{array}{l} z'[x] + y[x] + \big(z[x]\big)^3 = 0 \\ y'[x] + y^2[x] = 2z[x] \\ z[0] = y[0] = 1 \end{array} \right\rangle$$

In this case. Also, in case we have integer values, for example $p1 = 1$ and $p2 = 1$ I give them the values $p1 = 0.999999$ and $p2 = 0.999999$ so that the procedure can be executed.

Solution

Off[FindRoot::bddir,FindRoot::jsing,FindRoot::precw,Power::infy,Infinity::indet,General::cvmit,General: :stop, ReplaceAll::reps,Divide::infy,FindRoot::lstol,General::munfl, General::stop, General::precw];

Program data

$n =$ Input [Number fractional equations $=$ ”];

$p_j =$ Input [”Values of $p_j =$ ”] ;

$a_j =$ Input [”Value of $a_j =$ ”] ;

$da =$ Input [”Interval of $x =$ ”];

$q =$ Input [”Value of approch $h = 1/q =$ ”];

$u =$ Input [”Value of t to search for, $u =$ ”];

$c =$ Input [”Initial value for find Root $y^j_{x_i} =$ ”] ;

$y^j_{x_i}[0] =$ Input[”Initial value for $y^j_{x_i}[0] =$ ”] ;

Data of program

p1:=0.999999;p2:=0.999999;p 2 := $p4 = 0$;

$a3 := p3 = 0; a1 := 1; a2 := 1; a3 := 0; a4 := 0;$

$q := 2000; da := 1; u := 3/5;$

$j = 0; t := 0; h := da/q;$

$y[0] := 1; A[1] := y[0];$

$r[0] := 1; B[1] := r[0];$

Data processing

$L\lceil v\_] := v\hat{}2 + v^* \left(a1/h^{p1} + a3/h^{p3}\right)$

$M[v1\_] := v1\hat{}3 + v1^* \left(a2/h^{p2} + a4/h^{p4}\right);$

For $[i = 1, i <= q, i++,$

$t = i/q$

$j=j+1$

$y[i-1]=A[j];$

$r[i-1]=B[j]$

$$w[i]=N\left[F_1[t]+2r[i-1]-\sum_{j=1}^{2}\left(\frac{\alpha_j^1}{h^{p_j}}\sum_{k=1}^{i}\frac{\Gamma\left(k-p_j^1\right)}{\Gamma\left(p_j^1\right)\times\Gamma(k+1)}y(i-k)\right),10\right];$$

$s=N[z1]/.\text{FindRoot}\ [L[z1]-w[i]==0,\{z1,c\},\ \text{WorkingPrecision}\to 15];$

$y[i]=N[s,10]$

$$z[i]=N\left[F_2[t]-y[i]-\sum_{j=1}^{2}\left(\frac{\alpha_j^2}{h^{p_j^2}}\sum_{k=1}^{i}\frac{\Gamma\left(k-p_j^2\right)}{\Gamma\left(p_j^2\right)\times\Gamma(k+1)}r(i-k)\right),10\right];$$

$s1=N[z2]/.\text{FindRoot}\ [M[z2]-z[i]==0,\{z2,c\},\ \text{WorkingPrecision}\to 15];$

$r[i]=N[s1,10];$

$A[j+1]=N[y[i],10]$

$B[j+1]=N[r[i],10]$

$ta:=u/da;$

$If[t>=\text{ta}\,\&\&t<ta+1/q,$

Print["y[",N[t*da],"]=",N[y[i],10],,",","r[",N[t*da $\quad ,t^*]=",N[r[i],15]]\,,Loopback]]$

Design of solution functions

data 1 = Array$[y,\{q\},1]$;

FF1=Max [data 1];

data 2 = Array$[r,\{q\},1]$;

FF2=Max [data 2];

ListPlot [Table $[y[i],\{i,1,q,1\}]$, PlotRange $\to\{0,FF1+.8\}$, DataRange $\to\{0,1+1/20\}$, AxesStyle $\to$ Arrowheads $[\{0,0.03\}]$, AxesLabel $\to$ { "x","y(x)" }]

ListPlot [Table $[r[i],\{i,1,q,1\}]$, PlotRange $\to\{0,FF2+.8\}$,DataRange $\to\{0,1+1/20\}$, AxesStyle $\to$ Arrowheads $[\{0,0.03\}]$, AxesLabel $\to\{"x","z(x)"\}$]

ListPlot[{Labeled[Table[y[i], $\{i,1,q,1\}$], "y(x)"], Labeled[Table[r[i], {i, 1, q, 1}], "z(x)"]},PlotRange $\to$ $\{-0.2$, cc +.8}, DataRange $\to\{0,1+1/20\}$, AxesStyle $\to$ Arrowheads $[\{0,0.03\}]$,
AxesLabelAxesLabel $\to$ { "x", "z""(""x" ")""/""y""(""x"")" }]

Solution

Using the program after choosing approximation $h=1/2000$ and $p1=0.95$ and $p2=0.90$ we find for the functions at in $xi=0.6$ that the values are

$$y[0.6]=0.6120110924, z[0.6]=-0.04183256892$$

The diagrams concerning the solutions of the functions respectively will be

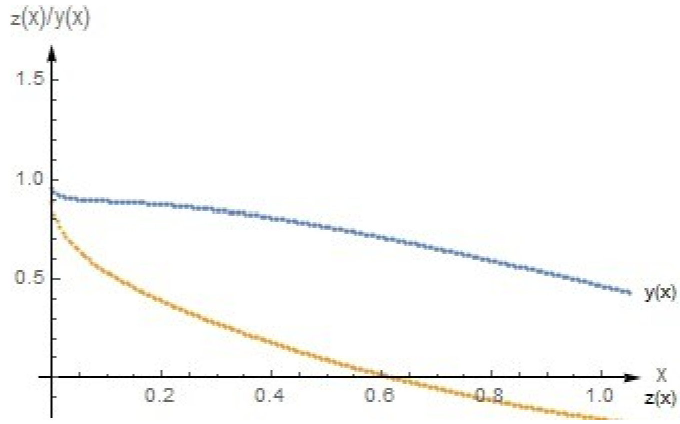

Figure 5:

While for the values $p1 = 1 \& p2 = 1$ we use the values $p1 = 0.999999 \& p2 = 0.999999$ and approximation $h = 1/2000$ and the solution will be for the value $Xi = 0.6$ for the 2 functions namely that:

$$y[0.6] = 0.9859672879, z[0.6] = 0.203829819$$

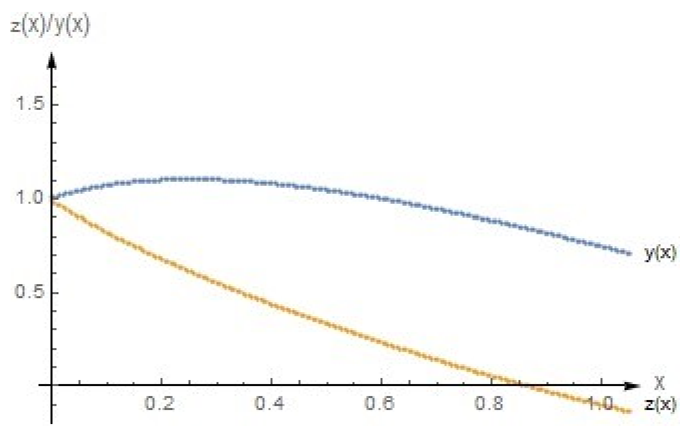

Figure 6:

Example 2. We have the case of the system

$$\left\langle \begin{array}{l} y[x] + y'[x] - (2z[x]) = 0 \\ z'[x] + z[x] = -y^{(1/2)}[x] - y[x] \\ z[0] = y[0] = 1 \end{array} \right\rangle$$

Also, in this case we have integer values, for example $p1 = 1$ and $p2 = 1$ and $p3 = 1/2, a4 = p4 = 0, a1 = a2 = a3 = 1$. I give them the values $p1 = 0.999999$ and $p2 = 0.999999$ so that the procedure can be executed using the program with approximation $h = 1/1000$ and $p1 = 0.999999, p2 = 0.999999$ and $p3 = 1/2$ we find for the functions at in $xi = 0.6$ that the values are

Solution

$$y[0.6] = 1.568491662z[0.6] = 1.388978643$$

The diagrams of the solutions of the functions respectively will be

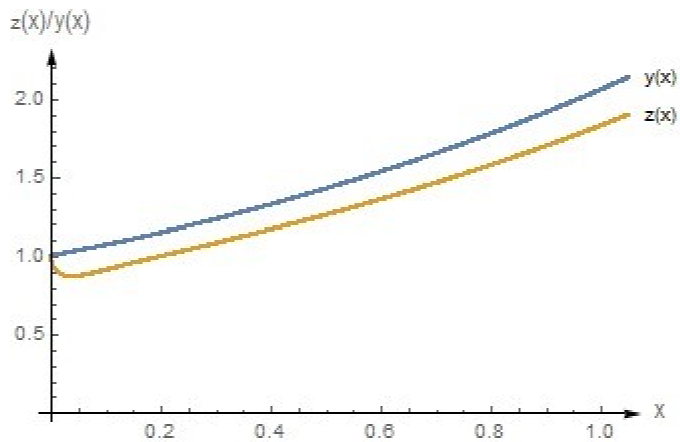

Figure 7:

XVI. Solving differential equations of integer degree by enhanced Euler method.

An initial value problem (IVP) is a differential equation with an initial condition that specifies the value of an unknown function at a specific point in the domain. Differential equations are commonly used in physics, chemistry, biology, and economics in science and engineering to solve a variety of physical problems.

Consider a first-order ordinary differential equation with initial value problem apply

$\frac{dy}{dx} = f(x, y)$ subject to initial condition $y(x_0) = y_0$.

The simplest and most known numerical method to solve is Euler's method.

$$y_{n+1} = y_n + hf(x_n, y_n), \ n = 0, 1, 2, \ldots \tag{VII.1}$$

The next stage of objection is

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} \cdot f(x_n, y_n)\right), \ n = 0, 1, 2, \ldots \tag{VII.2.}$$

If we extend it to 4 times enhanced process, we'll get the final relationship we're interested in

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} \cdot f\left(x_n, y_n + \frac{h}{2} f\left(x_n, y_n + hf\left(x_n, y_n f\left(x_n, y_n\right)\right)\right)\right)\right), \ n = 0, 1, 2, \ldots \tag{VII.3}$$

which uses the slope of the tangent to the mean, but y value is updated twice compared to the previous order. It is considered the most advantageous of other methods such as Picard or Kuta - Simson in terms

of programming command economy. Of course, with this method we can normally define the initial values in any class of simple differential and extra systems according to existing theory.

XVII. Examples and Programs for finding numerical approximations solutions for a differentials equations or system, in mathematica language.

XVII.1 Differential equation of order n

Example 1.

We used enhanced Euler method to approximate the Differential equation $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 4y$ in $y(0.1)$ when for $x = 0$ given $y(0) = 0.2$ and $y'(0) = 0.5$.

Solution

From the beginning we need to define the functions for first derivative and second derivative and then order the iterative program.

i) We therefore define for **the first derivative** the function $F(x, y, z) = y'(x) = z$

ii) We also define **the second derivative** as $G\left(x_2, y_2 z\right) = y''(x) = 4y(x) - 2xz$

Let's look at the solution of this differential equation with a program, choosing an approximation step $h = 0.0001$ then we will have:

Off[Array::ilsmn,ListPlot::prng];

$w := 0.2; j := 0.5; t := 0; h := 0.0001; q1 = 1/h; B[1] := 0.2; A[1] := 0.5$

$F[x, y, z] := z$

$G[x, y, z] := 4y - 2x * z$

$For[k = 1, k <= q1, k + +$

$w = w + h * F[t + h/2, w + h/2 * F[t, w + h/2 * F[t, w + h/2 * F[t, w + h * F[t, w, j], j], j], j], j];$

$j = j + h * G[t + h/2, w + h/2 * G[t, w + h/2 * G[t, w + h/2 * G[t, w + h * G[t, w, j], j], j], j], j]$

$t = t + h;$

$B[k] = w;$

$A[k] = j$

$If\left[t >= 0.1 \&\& t < 0.1 + 1/q1, \text{Print}\left[t, ", " N[w], ^{-1}, " N[j]\right], Loopback\right]\right]$

data $= \text{Array}[B, \{q1\}, 1]$

$FF1 = \text{Max}[\text{data}\,1]$

data 2 $= \text{Array}[A, \{q1\}, 1]$

FF2=Max [data 2];

cc $= \text{Max}[FF1, FF2]$

ListPlot[Table $[B[i], \{i, 1, q1, 1\}]$, PlotRange $\to \{0, FF1 + .2\}$,DataRange $\to \{0, 1 + 1/20\}$, AxesStyle $\to$ Arrowheads $[\{0, 0.03\}]$, AxesLabel $\to$ { "x","y(x)" }];

ListPlot[{Labeled[Table[B[i],{i,1,q1,1}],"y(x)"],Labeled[Table[A[i],{i,1,q1,1}],"dy/dx"]},
PlotRange $\to \{-0.2, cc + .8\}$,
DataRange $\to \{0, 1 + 1/20\}$, AxesStyle $\to$Arrowheads $[\{0, 0.03\}]$, AxesLabel $\to$ {"x ","dy/dx /$y(x)$"}]

The solution will be for the value $Xi = 0.6$ for the 2 functions namely that:

$$y[0.1] = 0.254164, y'[0.1] = 0.585019$$

The diagram of the solutions of the functions $y(x)$ and y '(x) respectively will be

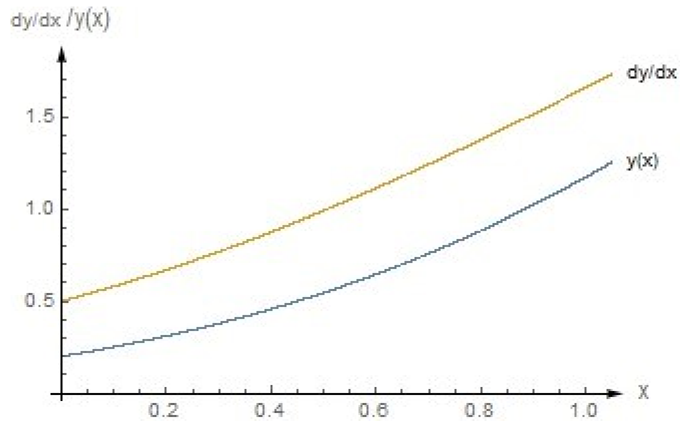

Figure 8:

Example 2.

Using the Euler enhanced method to approximate the differential equation $\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 3x^2 + 5$ at $y(0.1)$ when for $x = 0$ given $y(0) = 0.2$ and $y'(0) = 0.5$ and $y''(0) = 0.5$

Solution

i) We define for **the first derivative** the function $F(x, y, z, m) = y'(x) = z$
ii) Also we define for **the second derivative** as $G(x, y, z, m) = y$ " $(x) = m$
iii) And final we define for **the third derivative** as $H(x, y, z, m) = y'''(x) = 3x\hat{}2 + 5 - y - 2xz - xm$

Let's at the solution of this differential equation with a program in mathematica, choosing an approximation step $h = 0.0001$ then we will have:

Off[Array::ilsmn,ListPlot::prng];

$w := 0.2; j := 0.5; q := 0.7; t := 0; h := 0.0001; qw := 1/h;$

$F\left[x\_, y\_, z\_, m\_\right] := z;$

$G\left[x\_, y\_, z\_, m\_\right] := m;$

$H\left[x\_, y, z\_, m\_\right] := 3x\hat{}2 + 5 - y - 2x * z - x^*m$

For $[k = 1, k <= qw, k + +$

$w = w + h * F[t + h/2, w + h/2 * F[t, w + h/2 * F[t, w + h/2 * F[t, w + h * F[t, w, j, q], j, q], j, q], j, q], j, q]$

$j = j + h * G[t + h/2, w + h/2 * G[t, w + h/2 * G[t, w + h/2 * G[t, w + h * G[t, w, j, q], j, q], j, q], j, q], j, q]$

$q = q + h * H[t + h/2, w + h/2 * H[t, w + h/2 * H[t, w + h/2 * H[t, w + h * H[t, w, j, q], j, q], j, q], j, q], j, q]$

$r = H[t, w, j, q]$

$A[k] = w;$

$B[k] = j;$

$L[k] = q;$

$M[k] = r;$

$t = t + h;$

If $[t >= 0.1\&\&t < 0.1 + 1/qw,$

Print[t,”,”,N[w],”,”,N[j],”,”,N[q],”,”,N[r]]];

Loopback]

data1=Array[A, $\{q1\}, 1]$;

FF1=Max[data1];

data2=Array[B, {q 1}, 1];

FF2=Max[data2];

data3 = Array[L, $\{q1\}, 1]$;

FF3=Max[data3];

data4=Array[M, {q1},1];

FF4=Max[data4];

cc=Max[FF1,FF2,FF3,FF4];

ListPlot[{ Labeled[Table[A[i],{i,1,q1,1 }],”$y(x)$”], Labeled[Table[B[i], $\{i, 1, q1, 1\}$],”$y(x)$”],
Labeled [Table[L [i], {i,1,q1,1}],”y”(x)”],Labeled[Table[M[i],{i,1,q1,1 }], ”y”’(x)”] },
PlotRange $\to \{-0.2, cc + .8\}$, DataRange $\to \{0, 1 + 1/20\}$, AxesStyle $\to$ Arrowheads $[\{0, 0.03\}]$,
AxesLabel $\to$ { ”x”,”y(x)” }]

The solution will be for the value $x = 0.1$ for the 4 functions namely that:

$$y[0.1] = 0.254284, y'[0.1] = 0.593577, y''[0.1] = 1.16768, y'''[0.1] = 4.54041$$

The diagram of the solutions of the functions $y(x)$ and $y'(x), y''(x)$ and $y'''(x)$ respectively will be

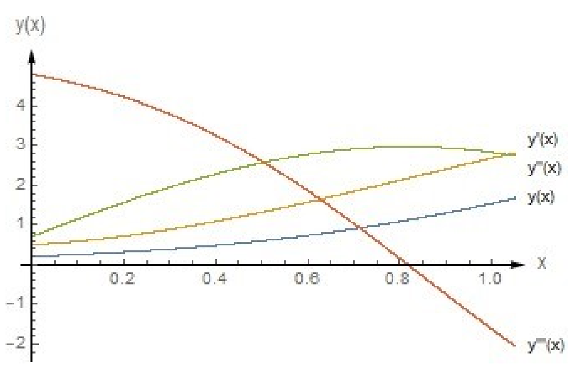

Figure 9:

XVII.2 Solving a system of differential equations

Example 3.

As an example we will deal with system of ordinary linear differential equations with constant coefficients, such as Using the Euler enhanced method to approximate the differential equation

$$2\frac{dx}{dt}+\frac{dy}{dt}-4x-y=e^{t}$$

$$\frac{dx}{dt}+3x+y=0$$

at $y(0.1)$ when for $x=0$ given $y'(0)=0.2$ and $x'(0)=0.5$

Solution

i) We define for **the first derivative** of y the function $F(t,y,x)=y'(t)=\mathrm{Exp}[t]+4x+4y+2(3x+y)$

ii) Also we define for **the first derivative** of x as $G(t,y,x)=x\text{ ' }(t)=-3x-y$

Off[Array::ilsmn,ListPlot::prng];

$w=0.2; j:=0.5; t=0; h:=0.0001; wq:=1/h;$

$F[t,y,x]:=\mathrm{Exp}[t]+4x+4y+2(3x+y)$

$G[t,y,x]:=-3x-y;$

For $[k=1,k<=wq,k++,$

$w=w+h*F[t+h/2,w+h/2*F[t,w+h/2*F[t,w+h/2*F[t,w+h*F[t,w,j],j],j],j],j];$

$j=j+h*G[t+h/2,w+h/2*G[t,w+h/2*G[t,w+h/2*G[t,w+h*G[t,w,j],j],j],j],j];$

$A[k]=w;$

$B[k]=j;$

$t=t+h;$

If $[t>=0.8\&\&t<0.8+1/wq,$

$qx=N\left[\left(e^{t}/8\right)-w/8-(3j)/4\right];$

$qy=-\left(\left(3e^{t}\right)/8\right)+(3w)/8+(5j)/4;$

$L[k]=qx;$

$M[k]=qy;$

Print$[t,",",N[w],",",N[j],",",N[qx],",",N[qy]]]$; Loopback];

data 1 = Array$[A,\{q1\},1]; FF1=$ Max[data 1];

data2 = Array $[S,\{q1\},1]; FF2=$ Max[data2];

data3 = Array $[L,\{q1\},1]; FF3=$ Max[data3];

data = Array[M, q 1$\},1]; FF4=$ Max[data4];

cc1=Max[FF1,FF2,FF3,FF4];

cc2=Min[FF1,FF2,FF3,FF4];

ListPlot[{ Labeled[Table $[A[i],\{i,1,q1,1\}]$,"$y(t)$ "], Labeled[Table[B[i],{i,1,q1,1}],"y'(t)"], Labeled[Table[L [i], $\{i,1,q1,1\}]$,"x "(t)"], Labeled [Table[M[i], {i,1,q1,1 }], ,"x"'(t)"] },
$PlotRange\rightarrow\{-\mathrm{cc}\,2,cc1+.8\}$, DataRange $\rightarrow\{0,1+1/20\}$, AxesStyle $\rightarrow$ Arrowheads $[\{0,0.03\}]$,
AxesLabel $\rightarrow$ { "x","y(x)" }]

The solution will be for the value $x = 0.1$ for the 4 functions namely that:

$$y[0.8] = 56.8435, x[0.8] = -7.15362, x'[0.8] = -1.46203, y'[0.8] = 11.5397$$

The diagram of the solutions of the functions $y(t)$ and $y'(t)$, $x(t)$ and $x'(t)$ respectively will be

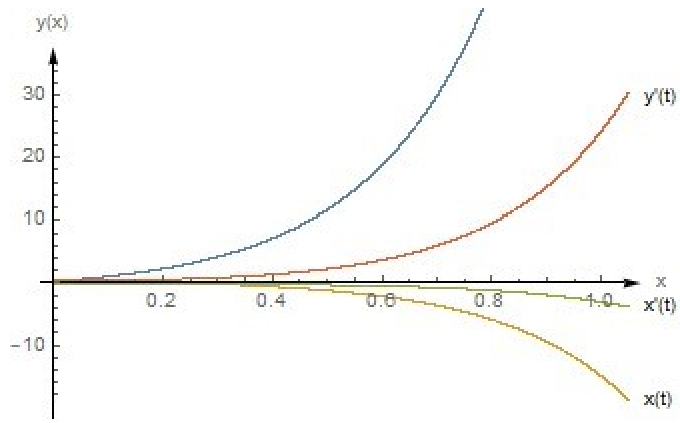

Figure 10:

Epilogue

With this paper we have put fractional differential equations and systems as well as differential equations and systems of integral degree on an approximate basis. Fractional calculus was born from a letter sent by L'Hospital to Leibniz on 30/09/1695 asking whether in a differential of degree n what happens if the degree is a fraction. The applications are enormous in physics, medicine, chemistry, astrophysics and it is possible to compute lateral values of quantities bounded between differentials of integer degree.

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